MINIMUM SENSITIVITY
LINEAR STOCHASTIC REGULATORS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

By

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Electrical Engineering

Georgia Institute of Technology

April, 1975

MINIMUM SENSITIVITY

LINEAR STOCHASTIC REGULATORS

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Date approved by Chairman: April 2, 1975

ACKNOWLEDGMENTS

This dissertation is dedicated to my parents, aunts, brother and sisters, whose love, support and sacrifices made possible my success in academic life, to my teachers, and to my wife for her love, encouragement and patience.

I wish to express my sincere gratitude to Professor R. P. Webb for his advice and guidance, without which completion of this dissertation would not have been possible. I also wish to thank Drs. C. Y. Chong and M. E. Womble for their services as members of the reading committee.

Special appreciation is given to the Civil Service Commission of the Royal Thai Government for the scholarship which made possible my doctoral studies. Also, many thanks go to Sathaporn Petchrawises whose help allowed me more time to concentrate on my research.

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ABBREVIATIONS, CONVENTIONS AND NOTATIONS

Abbreviations

D.E.	differential equation
LDR	linear deterministic regulator
LSR	linear stochastic regulator
MSES	mean square estimate sensitivity
MSS	mean square sensitivity
MSTS	mean square trajectory sensitivity

Conventions

$A(i,j)$, A_{ij}	ij^{th} element of a matrix A
$E\{x(t)\}$	the expectation of a stochastic process $x(t)$
$tr\{A\}$	trace of a square matrix A
$x_i(t)$	i^{th} component of a vector $\underline{x}(t)$

Subscript $\alpha(\alpha_i)$ denotes a partial derivative with respect to $\alpha(\alpha_i)$.

Variables without the arguments indicating explicitly otherwise are understood to be nominal variables.

For symmetric matrices, occasionally only the upper triangular elements are shown

Notations

A	state weight matrix of J
A_1, A_2, A_3, A_4, A_5	weight matrix associated with J_M
A_6, A_7, A_8	weight matrix associated with J_M'
B	control weight matrix of J
B_1, B_2, B_3	weight matrix associated with J_K, J_K' and $\hat{J}_K$
C, C_a	deterministic controller gain of an LDR or an LSR
C_α	sensitivity of C to a scalar parameter α
C_{α_i}	sensitivity of C to a parameter component α_i
D	coefficient matrix of a linear D.E.
D_2	output feedback gain
E	the expectation operator
F	state coefficient matrix
G	control coefficient matrix
H	measurement matrix
H	the Hamiltonian
J	overall cost of an LSR
J_0	nominal overall cost of an LSR
$J(W_A), J(W_B), J(W_C)$	nominal overall costs of a compensated system corresponding to weight sets W_A, W_B and W_C, respectively
J_d	overall cost of an LDR
$J_M, J_M', J_K, J_K', \hat{J}_K$	generalized overall cost of an LSR
J_s	sensitivity performance index

K	Kalman gain
K_α	sensitivity of a Kalman gain to a scalar parameter α
K_{α_i}	sensitivity of K to a parameter component α_i
M	weight matrix associated with J_s
P	state-estimate error covariance matrix
P_0	initial value of P
$P_{xy}(t)$	cross-correlation of $\underline{x}(t)$ and $\underline{y}(t)$, i.e., $E\{\underline{x}(t)\underline{y}^T(t)\}$, for arbitrary $\underline{x}(t)$ and $\underline{y}(t)$
$P_{\theta\theta}(t)$	auto-correlation of $\underline{\theta}(t)$ of a compensated system, i.e., $E\{\underline{\theta}(t)\underline{\theta}^T(t)\}$
$P^i_{\theta\theta}(t)$	auto-correlation of $\underline{\theta}^i(t)$ of a compensated system, i.e., $E\{\underline{\theta}^i(t)\underline{\theta}^{iT}(t)\}$
$P_{\nu\nu}(t)$	auto-correlation of $\underline{\nu}(t)$ of an original LSR, i.e., $E\{\underline{\nu}(t)\underline{\nu}^T(t)\}$
$P^i_{\nu\nu}(t)$	auto-correlation of $\underline{\nu}^i(t)$ of an original LSR
$P_{\zeta_i\zeta_i}(t)$	mean square sensitivity of $x_i(t)$ of an original LSR to a scalar parameter α
$P^j_{\zeta_i\zeta_i}(t)$	mean square sensitivity of $x_i(t)$ of an original LSR to a parameter component α_j
$P_{\xi_i\xi_i}(t)$	mean square sensitivity of $x_i(t)$ of a compensated system to a scalar parameter α
$P_{\xi_i\xi_i}(W_A), P_{\xi_i\xi_i}(W_B), P_{\xi_i\xi_i}(W_C)$	$P_{\xi_i\xi_i}(t)$'s corresponding to weight sets W_A, W_B, and W_C, respectively
$P^j_{\xi_i\xi_i}(t)$	mean square sensitivity of $x_i(t)$ of a compensated system to a scalar parameter component α_j

$P^j_{\xi_i\xi_i}(W_A), P^j_{\xi_i\xi_i}(W_B), P^j_{\xi_i\xi_i}(W_C)$	$P^j_{\xi_i\xi_i}(t)$'s corresponding to weight sets W_A, W_B, and W_C, respectively
Q_f, Q_1, Q_2, Q_3, Q_5	weight matrix associated with J_s
Q_0	initial value of $P_{\theta\theta}$ or $P_{\nu\nu}$
Q_v	measurement noise covariance
Q_w	process noise covariance
R	weight matrix associated with J_s
S	compensating feedback gain
S_2, S_a	Riccati gain
$S_{2\alpha}$	sensitivity of S_2 to a scalar parameter α
S_f	weight matrix for $\underline{x}(t_f)$ associated with J
u	control variable
u_d^*	optimal nominal control of an LDR
v	measurement noise
W_A, W_B, W_C	weight set associated with J_s
w	process noise
x	plant state
x_d^*	optimal nominal state of an LDR
z	measurement
α	parameter
α_n	nominal value of α
δ	prefix to denote a small increment
$\delta(t)$	delta function
ζ	trajectory sensitivity of an original LSR
η	residual or innovation process

θ	augmented state of a compensated system
Λ	costate matrix corresponding to $P_{\theta\theta}$
λ	costate corresponding to mean variable μ_θ
μ_θ	mean value of arbitrary $\theta(t)$
ν	augmented state of an original LSR
ξ	trajectory sensitivity of a compensated system
σ	pseudo trajectory sensitivity of a compensated system
ω	stochastic process

SUMMARY

A linear stochastic regulator (LSR) is a cascade connection of an actual plant to be controlled, a deterministic controller and a Kalman filter. Since the controller and the filter are designed on the basis of the nominal parameter values which in general differ from the actual parameter values, the LSR is subject to the sensitivity problem caused by the parameter mismatches.

The objective of this research is to perform a trajectory sensitivity analysis, to obtain a new compensation method that will effect a reduction in the trajectory sensitivity of an LSR, to demonstrate the effectiveness of the method via simulation examples, and to compare the compensation method with other existing compensation methods.

The compensation method resulting from this research is an off-line technique based on sensitivity concepts, with a two-degree-of-freedom control structure in the form of a combined open-loop and closed-loop control. The feedback path contains a compensating gain which is chosen to minimize a stochastic integral performance index containing quadratic terms in trajectory sensitivity functions. The main task involved in the compensation method is to obtain the optimal compensating gain. The stochastic integral performance index is converted to

its deterministic equivalent. The Matrix Minimum Principle is then applied to the resulting deterministic problem to obtain the necessary conditions for calculation of the optimal compensating gain. The gain is obtained using a steepest-descent gradient algorithm. Four examples are examined to demonstrate the effectiveness of the compensation method.

The compensation method has two distinct advantages over other methods. Firstly, it enables a reduction in the trajectory sensitivity to be effected independently of the system trajectory; thereby the specifications on both the trajectory sensitivity and the system trajectory can be simultaneously met. Secondly, it causes the compensated system to reduce on the average to the original LSR when the parameter variations are zero. Another advantage, rendered by the chosen control structure, is that in many cases it results in only a nominal increase in the overall cost.

The compensation method, like other comparable methods, results in a two-point boundary value problem with a fairly large number of equations. Therefore, its uses may be limited to low-order systems. However, a procedure has been outlined that will probably extend its uses to fourth- or fifth-order systems. For higher-order systems, a suitable model order reduction technique can be used to first reduce the order of the systems for application of the compensation method.

CHAPTER I

INTRODUCTION

The purpose of the research is to study the trajectory sensitivity due to parameter variations or modelling errors of a linear stochastic regulator (LSR), and to obtain a new compensation method that will effect a reduction in the system trajectory sensitivity. The trajectory-sensitivity differential equations for the LSR are derived. The compemsation method is developed, shown via simulation examples to reduce the trajectory sensitivity of LSR's, and compared with the relevant existing methods.

Definition of the Problem

Linear regulators are applicable to a number of important practical problems, for example, autopilot designs, process controls, and many other control applications. Linear regulators are divided into two classes. One class is referred to as linear deterministic regulators (LDR's) in which there are no stochastic disturbances acting on the systems. Another is a more general class, known as linear stochastic regulators (LSR's), in which stochastic disturbances are incorporated in the plant dynamics and in the measurement systems. An LDR consists of an actual plant with the actual parameters $\underline{\alpha}$ and a deterministic controller with gain C(t).

They are connected in a closed-loop configuration in which the plant states are fed back, via the deterministic controller, to the plant input. An LSR is very similar to an LDR except that a Kalman filter with gain K(t) is used to obtain the estimates of the plant states and then the estimates are fed back to the plant input via a deterministic controller with gain C(t). In practice, the gains, C(t) and K(t), must be calculated on the basis of the nominal values of the parameters $\underline{\alpha}$, $\underline{\alpha}_n$, because the actual values of the parameters $\underline{\alpha}$ are unknown; only the approximate values of $\underline{\alpha}$, i.e., $\underline{\alpha}_n$, are known through measurements, identifications, educated guesses, and so on. When these gains, calculated on the basis of $\underline{\alpha}_n$, are used in the linear regulators to control plants with actual parameter values $\underline{\alpha}$, the performance characteristics of the linear regulators will certainly deviate from the design (nominal) characteristics. The performance characteristic of interest to the current research is the system trajectory sensitivity, or equivalently, deviations of the actual system trajectory from the nominal trajectory under the influence of parameter variations.

Historical Background of the Problem of Trajectory Sensitivity Reduction

The essence of sensitivity analysis is to consider the deviation of a system from its nominal behaviour caused by deviations of the system components and parameters from their nominal values. Sensitivity questions arise whenever one tries to construct a physical system from a set of mathematical specifications. A sound system design should incorporate sensitivity considerations to ensure that the system sensitivity to parameter variations remains within a permissible limit, or else the deviation of the actual system performance from the nominal may be beyond the permissible limit. Bode [21] was the first to introduce parameter sensitivity of control systems in his book in 1945. The idea of sensitivity received very little attention until 1955 when work began to appear using Bode's definition in relation to other system characteristics. During the years 1957 through 1962, sensitivity analysis received an increased attention; many authors related system stability and system characteristics to sensitivity functions.

The sensitivity concept had not been applied to the design of optimal control systems until 1963 when Dorato [23] introduced the idea of performance index sensitivity and showed that it could be useful in the design of such systems. In 1965, Siljak and Dorf [24] introduced a general performance index, which included both the sensitivity and performance characteristics, to derive an optimal control claimed to accomplish a reduction in sensitivity as well as optimization of the performance index. A great deal of research in the area of trajectory sensitivity reduction had been carried out for LDR's. Many authors (D'Angelo et al. 1966 [28], Tuel et al. 1966 [30], Cassidy and Lee 1967 [26], Dougherty et al. [27], Kreindler 1967 [29], Bradt 1968 [25], Higginbotham 1969 [34]) attempted to reduce the trajectory sensitivity of LDR's using a general performance index similar to that introduced by Siljak and Dorf. In these attempts, the sensitivity reduction achieved was a trade-off between the system sensitivity characteristics and the system trajectory characteristics; and furthermore the compensated systems did not reduce to the original (uncompensated) nominal systems when there were no parameter variations present. Grübel and Kreisselmeir [32] introduced, for a linear deterministic control system, a control system with a two-degree-of-freedom control structure that enabled minimization of trajectory sensitivity to be effected independently of the nominal trajectory, and that caused the compensated system to reduce to the original system when the parameter variations were zero. Relatively little work has been done to reduce the trajectory sensitivity of an LSR. Higginbotham [35] attempted to reduce the trajectory sensitivity of an LSR. But he considered only the modelling errors in the deterministic controller while leaving the modelling errors in the Kalman filter uncompensated. This might be inadequate because any modelling errors in the Kalman filter would cause the filter to give an erroneous state estimate (it had been established in [10] and [12] that in pure estimation applications modelling errors were one of the main factors that caused divergence in the Kalman filters in the sense that the actual state estimates were subject to very large errors). When the erroneous state estimate was fed back as the plant input, the resulting system trajectory would certainly be off the nominal. Stavroulakis and Sarachik [50] also considered the problem of reducing trajectory sensitivity of an LSR. They considered modelling errors in both the deterministic controller and the Kalman filter. In their process of deriving the formulae they made an error which is discussed in Appendix C. Disregarding the error, they attempted to minimize the expected value of an integral performance index quadratic in the state, control and the state-estimate sensitivity. They included the state-estimate sensitivity instead of the state sensitivity in the performance index solely on the basis of their intuitive reasoning. It should be noted, however, that the state sensitivity, not the state-estimate sensitivity, is the factor one aims to minimize; and that there is no guarantee that a reduction in the state-estimate sensitivity will result in a corresponding reduction in the state sensitivity. Moreover, their approach does not permit a trajectory sensitivity reduction which is independent from the optimal nominal trajectory; and finally the compensated system does not on the average reduce to the original system when there are no modelling errors. The new compensation method to be developed in this research is based on a two-degree-of-freedom control structure which is extended for stochastic systems. It has all the desirable features in that modelling errors in both the deterministic controller and the Kalman filter are accounted for, that it is an off-line compensation technique, that the trajectory sensitivity minimization can be effected independently of the optimal nominal trajectory, that the compensated system does on the average reduce to the optimal nominal system when there are no modelling errors, and that the number of equations involved in the resulting two-point boundary-value problem (TPBVP) is comparable to that of the correct version of the method introduced by Stavroulakis and Sarachik.

Organizational Outline

The dissertation consists of six chapters and three appendices. In Chapter II, the mathematical formulations of an LSR and an LDR are given. The relationships between the LSR and the LDR, which are needed in the formulations of the compensation method, are developed. Also given is the sensitivity analysis of the original LSR which helps to determine which parameter causes the trajectory sensitivity of significant magnitude; and only the significant trajectory sensitivity should be compensated. Relationship between the trajectory sensitivity and the state-estimate sensitivity of the original LSR is also established.

In Chapter III, the problem statement is given and then it is followed by the mathematical formulations of the compensation method. At first, the compensation method is given for a multi-variable system with a scalar parameter, and then it is extended to a multi-parameter, multi-variable system. Inter-relationships between the compensated system and the original LSR, as well as the intra-relationships between various quantities of the compensated system, are derived. Utilizing these relationships, it is shown that a considerable saving in the computational burden in terms of a reduced number of equations can be achieved.

In Chapter IV, four numerical examples are given to demonstrate the usefulness, workability and effectiveness of the compensation method in reducing the trajectory sensitivity of the LSR's. And, where possible, the analytical solutions to the problems are attempted to gain some insight into the problems.

In Chapter V, the compensation method is compared with other existing methods; emphasis will be placed on those methods based on the sensitivity concept. The comparison is on the basis of the main features and the computational burden of the various methods.

Chapter VI presents the conclusions of the research. Appendix A gives detailed mathematical derivations of the equations for the sensitivity analysis of the original LSR for more general modelling errors than those given in Chapter

II. Appendix B deals with the details of mathematical derivations of the equations involved in the compensation method for general modelling errors. And finally, Appendix C accounts for the summary of some of the compensation methods with which the compensation method is compared.

CHAPTER II

LINEAR REGULATORS

In this chapter, the mathematical formulations of LDR's and LSR's are presented. Certain pertinent properties relating the LDR's to the corresponding LSR's are derived. Trajectory sensitivity analysis of the LSR's is performed by deriving a set of sensitivity differential equations, and some useful relationship between the trajectory sensitivity and the estimate sensitivity is also derived. The reason for inclusion of LDR's is that the research being undertaken utilizes certain properties of LDR's as well.

Linear Deterministic Regulators

An LDR is the problem of finding a control $\underline{u}(t)$ such that the performance index

$$J_d = \frac{1}{2}||x(t_f)||^2_{S_f} + \frac{1}{2}\int_{t_o}^{t_f}[||\underline{x}(t)||^2_{A(t)} + ||\underline{u}(t)||^2_{B(t)}]\,dt \quad (2.1)$$

is minimized, subject to

$$\dot{\underline{x}} = F(\underline{\alpha},t)\underline{x} + G(\underline{\alpha},t)\underline{u}(t),\ \underline{x}(t_o) = \underline{x}_o\,, \quad (2.2)$$

where $\underline{u}(t)$ is a control r-vector, $\underline{x}(t)$ is a state n-vector,

$A(t)$ and S_f are symmetric positive semi-definite. $n \times n$-matrices, $B(t)$ is a symmetric positive definite $r \times r$-matrix, $\underline{\alpha}$ is a parameter m-vector with the nominal value of $\underline{\alpha}_n$, t is an independent variable (usually time), and t_0 and t_f are fixed initial and final times, respectively. It can be shown (see [51], for example) that the deterministic optimal control, $\underline{u}_d^*(\underline{\alpha},t)$, is given by

$$\underline{u}_d^*(\underline{\alpha},t) = -C(\underline{\alpha},t)\underline{x}_d^*(\underline{\alpha},t) \tag{2.3}$$

where $\underline{x}_d^*(\underline{\alpha},t)$ is the corresponding deterministic optimal trajectory which satisfies

$$\dot{\underline{x}}_d^*(\underline{\alpha},t) = [F(\underline{\alpha},t) - G(\underline{\alpha},t)C(\underline{\alpha},t)]\underline{x}_d^*(\underline{\alpha},t), \quad \underline{x}_d^*(t_0) = \underline{x}_0, \tag{2.4}$$

where $C(t)$ is determined from (2.5) and (2.6) below.

$$C(\underline{\alpha},t) = B^{-1}(t)G^T(\underline{\alpha},t)S_2(\underline{\alpha},t), \tag{2.5}$$

$$\begin{aligned}\dot{S}_2(\underline{\alpha},t) = &-S_2(\underline{\alpha},t)F(\underline{\alpha},t) - F^T(\underline{\alpha},t)S_2(\underline{\alpha},t) + C^T(\underline{\alpha},t)B(5)C(\underline{\alpha},t) \\ &- A(t), \quad S_2(t_f) = S_f .\end{aligned} \tag{2.6}$$

$C(\underline{\alpha},t)$ and $S_2(\underline{\alpha},t)$ are $r \times n$ matrix and $n \times n$ symmetric matrix, respectively. $C(\underline{\alpha},t)$ is the gain of the deterministic controller.

Since in most cases the actual value of the parameter $\underline{\alpha}$ is unknown, the designer has to base all of his calculations on the nominal value of $\underline{\alpha}$, $\underline{\alpha}_n$, which is generally only an approximate value of the parameter $\underline{\alpha}$. That is, $\underline{\alpha} = \underline{\alpha}_n$ must be used in (2.2) - (2.6) which now become

$$\dot{\underline{x}}_d^* = F(\underline{\alpha}_n,t)\underline{x}_d^* + G(\underline{\alpha}_n,t)\underline{u}_d^*(t), \quad \underline{x}_d^*(t_o) = \underline{x}_o; \qquad (2.7)$$

$$\underline{u}_d^*(\underline{\alpha}_n,t) = -C(\underline{\alpha}_n,t)\underline{x}_d^*(\underline{\alpha}_n,t) \ ; \qquad (2.8)$$

$$\dot{\underline{x}}_d^*(\underline{\alpha}_n,t) = [F(\underline{\alpha}_n,t) - G(\underline{\alpha}_n,t)C(\underline{\alpha}_n,t)]\underline{x}_d^*(\underline{\alpha}_n,t), \underline{x}_d^*(t_o) = \underline{x}_o; \quad (2.9)$$

$$C(\underline{\alpha}_n,t) = B^{-1}(t)G^T(\underline{\alpha}_n,t)S_2(\underline{\alpha}_n,t); \qquad (2.10)$$

$$\dot{S}_2(\underline{\alpha}_n,t) = -S_2(\underline{\alpha}_n,t)F(\underline{\alpha}_n,t) - F^T(\underline{\alpha}_n,t)S_2(\underline{\alpha}_n,t)$$
$$+ C^T(\underline{\alpha}_n,t)B(t)C(\underline{\alpha}_n,t)-A(t), S_2(t_f) = S_f. \quad (2.11)$$

Equations (2.7), (2.8), and (2.9) define the nominal plant, the optimal nominal control $\underline{u}_d^*(\underline{\alpha}_n,t)$, and the optimal nominal trajectory $\underline{x}_d^*(\underline{\alpha}_n,t)$, respectively. Equations (2.10) and (2.11) define the nominal controller gain $C(\underline{\alpha}_n,t)$. The nominal LDR is shown in Figure 1. For convenience, the following convention will be used throughout this dissertation. Any variables without the arguments indicating explicitly otherwise will be understood to be the nominal variables, i.e., being evaluated at $\underline{\alpha} = \underline{\alpha}_n$.

In summary, an LDR is the problem defined by (2.1) and (2.2) and its solution is given by (2.3) - (2.6). Since the nominal parameter value $\underline{\alpha}_n$ has to be used as noted earlier, the designer has to work with the nominal plant

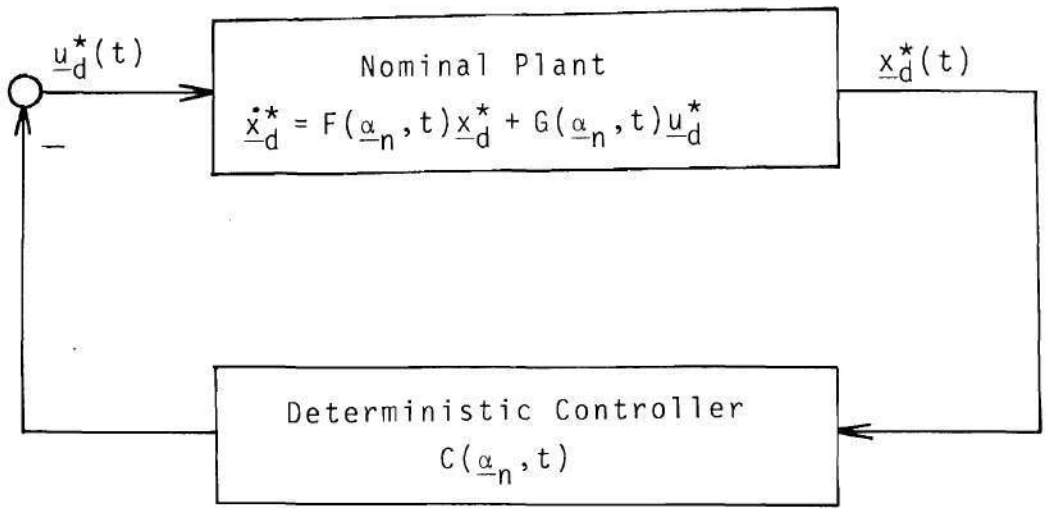

Figure 1. A Nominal Linear Deterministic Regulator

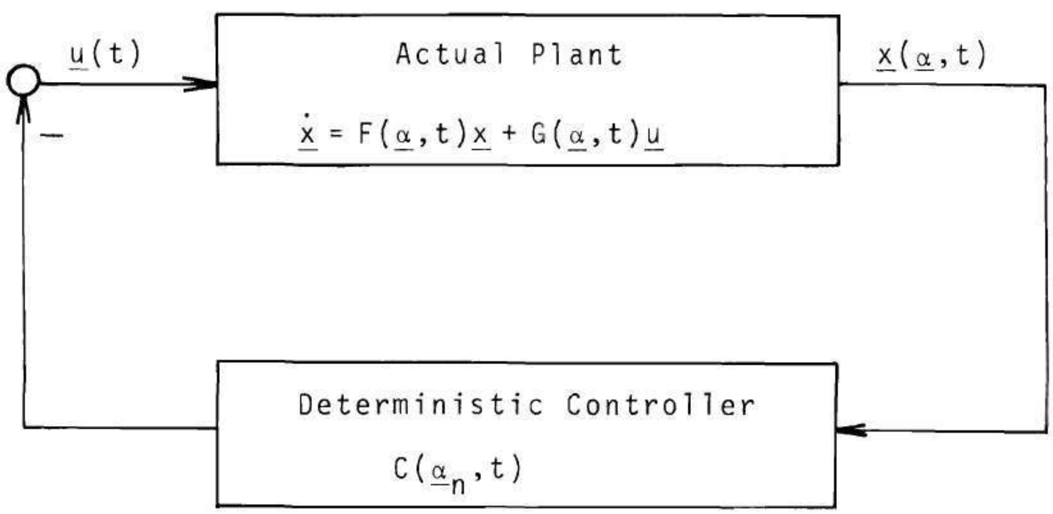

Figure 2. An Actual Linear Deterministic Regulator

in (2.7), and consequently with the nominal variables defined by (2.8) - (2.11). In a practical application, the actual LDR is as shown in Figure 2, where the nominal controller gain, $C(\underline{\alpha}_n,t)$, is used to control the actual plant whose parameter value is $\underline{\alpha}$, not $\underline{\alpha}_n (\underline{\alpha} \neq \underline{\alpha}_n$, in general); thereby the sensitivity problem arises and the actual plant state, $x(\underline{\alpha},t)$, will differ from the optimal nominal state, $\underline{x}_d^*(t)$.

Linear Stochastic Regulators

An LSR is a more general problem than an LDR in that the stochastic disturbances are incorporated into the system dynamics as and the measurement system additive independent white Gaussian noises. The mathematical description of an LSR is as follows. Find a control $\underline{u}(t)$ such that the performance index

$$J = E\ \{\frac{1}{2}\|\underline{x}(t_f)\|^2_{S_f} + \frac{1}{2}\int_{t_o}^{t_f} [\|\underline{x}(t)\|^2_{A(t)} + \|\underline{u}(t)\|^2_{B(t)}]\ dt\}, \quad (2.12)$$

is minimized subject to

$$\dot{\underline{x}} = F(\underline{\alpha},t)\underline{x} + G(\underline{\alpha},t)\underline{u}(t) + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o, \quad (2.13)$$

and the measurement

$$\underline{z} = H(t)\underline{x} + \underline{v}(t) \quad (2.14)$$

where $\underline{z}$ is a measurement l-vector; H(t) is an $l \times n$ measurement matrix; and $\underline{w}(t)$ and $\underline{v}(t)$ are the process and measurement white Gaussian noises, respectively, with the following properties:

$$E\{\underline{w}(t)\} = \underline{0} \ , \tag{2.15}$$

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = Q_w(t)\delta(t-\tau), \tag{2.16}$$

$$E\{\underline{v}(t)\} = \underline{0} \ , \tag{2.17}$$

$$E\{\underline{v}(t)\underline{v}^T(\tau)\} = Q_v(t)\delta(t-\tau), \tag{2.18}$$

$$E\{\underline{w}(t)\underline{v}^T(\tau)\} = 0 \ . \tag{2.19}$$

Other notations are the same as previously defined. It can be shown [52 - 53] that the optimal control is given by

$$\underline{u}(\underline{\alpha},t) = -C(\underline{\alpha},t)\hat{\underline{x}}(\underline{\alpha},t) \tag{2.20}$$

where $C(\underline{\alpha},t)$ and $\hat{\underline{x}}(\underline{\alpha},t)$ are the deterministic controller gain and the optimal state estimate, respectively. The controller gain, $C(\underline{\alpha},t)$, is exactly the same as that of the LDR, given by (2.5) and (2.6). The optimal state estimate is given by (2.21) - (2.23) below:

$$\dot{\hat{\underline{x}}} = F(\underline{\alpha},t)\hat{\underline{x}} + G(\underline{\alpha},t)\underline{u}(t) + K(\underline{\alpha},t)[\underline{z}(t)-H(t)\hat{\underline{x}}], \ \hat{\underline{x}}(t_o)=\underline{x}_o, \tag{2.21}$$

$$K(\underline{\alpha},t) = P(\underline{\alpha},t)H^T(t)Q_v^{-1}, \tag{2.22}$$

$$\dot{P}(\underline{\alpha},t) = F(\underline{\alpha},t)P(\underline{\alpha},t) + P(\underline{\alpha},t)F^T(\underline{\alpha},t) - K(\underline{\alpha},t)Q_v K^T(\underline{\alpha},t) \quad (2.23)$$

$$+ Q_w(t), \; P(t_0) = P_0.$$

However, the designer is faced with the same problem of the unknown actual value of the parameter $\underline{\alpha}$ as in the case of the LDR, and therefore he has to be contented with the nominal LSR given below by (2.24) - (2.29).

$$\dot{\underline{x}} = F(t)\underline{x} + G(t)\underline{u}(t) + \underline{w}(t), \; E\{\underline{x}(t_0)\} = \underline{x}_0; \quad (2.24)$$

$$\underline{z} = H(t)\underline{x} + \underline{v}(t); \quad (2.25)$$

$$\underline{u}(t) = -C(t)\hat{\underline{x}}(t); \quad (2.26)$$

$$\dot{\hat{\underline{x}}} = F(t)\hat{\underline{x}} + G(t)\underline{u}(t) + K(t)[\underline{z}(t) - H(t)\hat{\underline{x}}], \hat{\underline{x}}(t_0) = \underline{x}_0; \quad (2.27)$$

$$K(t) = P(t)H^T(t)Q_v^{-1}(t); \quad (2.28)$$

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) - K(t)Q_v K^T(t) + Q_w(t), \; P(t_0) = P_0. \quad (2.29)$$

The nominal LSR is shown in Figure 3. Equations (2.10) and (2.11) define what is well-known as the deterministic controller of which the gain is C(t). Equations (2.26) - (2.29) define the Kalman filter of which K(t) is known as the Kalman gain. An inspection of the equations governing the controller and the Kalman filter, reveals that they can be independently designed off-line and connected in cascade as shown in Figure 3. This is the

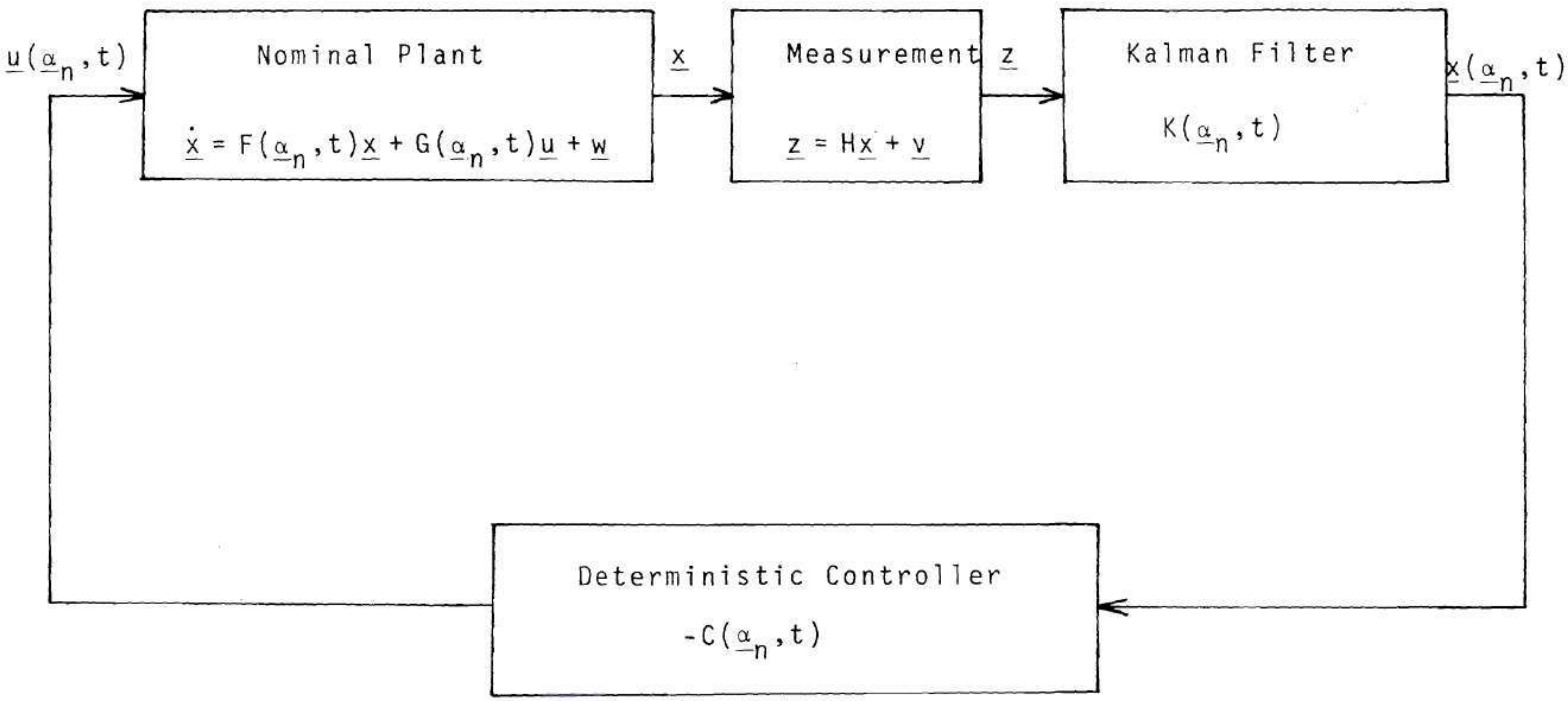

Figure 3. A Nominal Linear Stochastic Regulator.

consequence of the well-known Separation Theorem which states: "For a linear system with quadratic cost functional and subject to independent additive white Gaussian noise inputs, the optimal stochastic controller is realized by cascading an optimal estimator with a deterministic controller." In practice, the Kalman gain, K(t), and the controller gain, C(t), are pre-calculated off-line on the basis of the nominal parameter value $\underline{\alpha}_n$, i.e., using (2.10) and (2.11) for C(t) and (2.28) and (2.29) for K(t), (the actual parameter value, $\underline{\alpha}$, is not available) and then connected to the actual plant (with the actual parameter value $\underline{\alpha}$) to form an actual LSR as shown in Figure 4. Once gain, one has the sensitivity problem due to the parameter mismatches and the actual trajectory, $\underline{x}(\alpha,t)$, of the system in Figure 4 will differ from the optimal nominal system trajectory, $\underline{x}(\underline{\alpha}_n,t)$, of the system in Figure 3.

Relationships between LDR and LSR

There exist some useful relationships between a nominal LDR and the corresponding nominal LSR, which will be utilized in the research later on.

Property 2.1:

The mean optimal nominal trajectory (estimate) of a given LSR is the same as the deterministic optimal nominal trajectory of the corresponding LDR, i.e.,

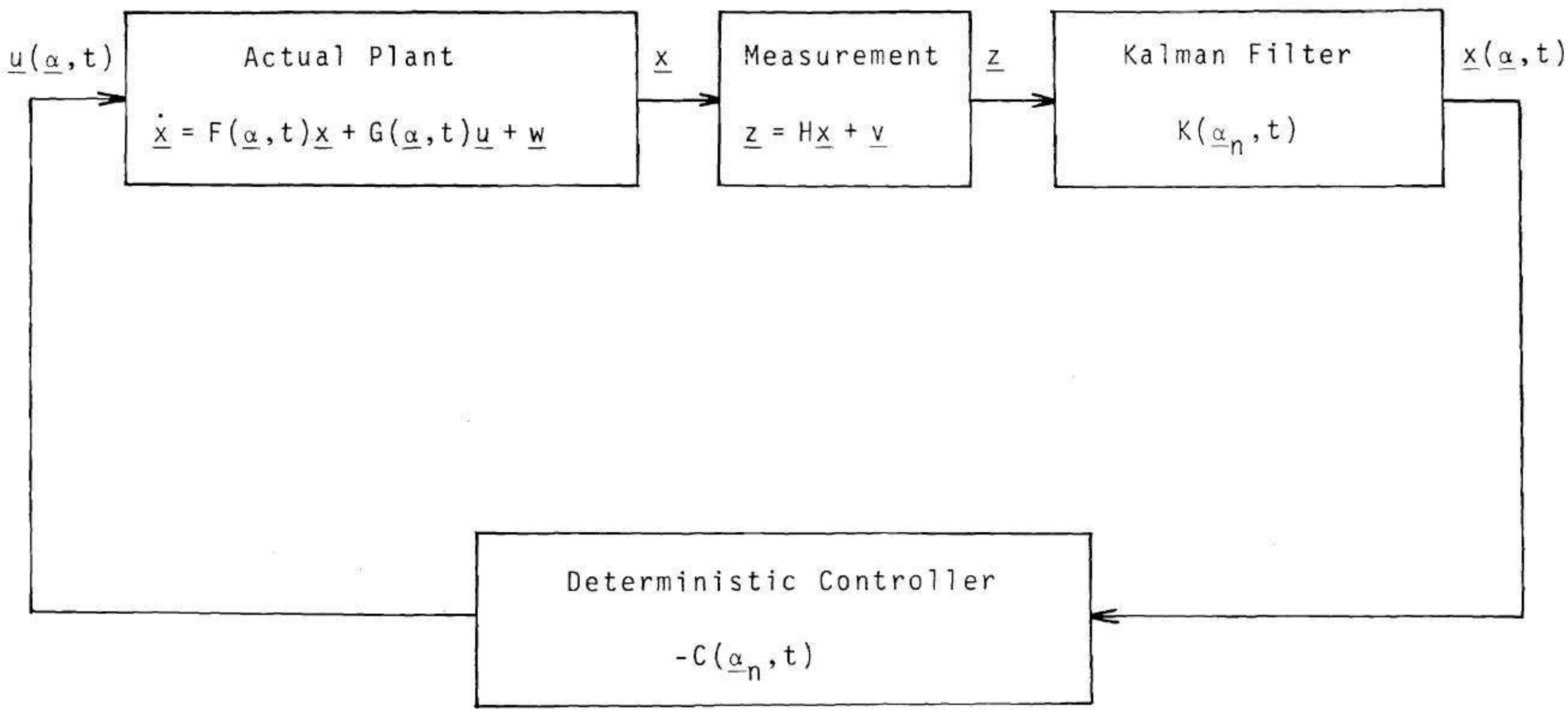

Figure 4. An Actual Linear Stochastic Regulator.

$$E\{\underline{x}(t)\} = \underline{x}_d^*(t) \ .$$

where $\underline{x}(t)$ is the optimal nominal trajectory of the LSR and $\underline{x}_d^*(t)$ is the deterministic optimal nominal trajectory of the LDR.

Proof:

Substituting $\underline{u}(t)$ from (2.26) into (2.24) and taking the expected value of both sides, one has

$$E\{\dot{\underline{x}}(t)\} = F(t)E\{\underline{x}(t)\} - G(t)C(\underline{\alpha}_n,t)E\{\hat{\underline{x}}(t)\},$$

$$E\{\underline{x}(t)\} = \underline{x}_0 \ . \tag{2.30}$$

Using the unbiased property of the Kalman filter:

$$E\{\underline{x}(t)\} = E\{\hat{\underline{x}}(t)\} \ , \tag{2.31}$$

equation (2.30) becomes

$$E\{\dot{\underline{x}}(t)\} = [F(t) - G(t)C(t)]E\{\underline{x}(t)\}, E\{\underline{x}(t_0)\} = \underline{x}_0 . \tag{2.32}$$

Comparing (2.32) with (2.9) reveals that

$$E\{\underline{x}(t)\} = \underline{x}_d^*(t) \ . \tag{2.33}$$

Also, using (2.31), (2.33) becomes

$$E\{\underline{x}(t)\} = E\{\hat{\underline{x}}(t)\} = \underline{x}_d^*(t) \ . \tag{2.34}$$

Property 2.2:

The mean optimal nominal control of a given LSR is the same as the deterministic optimal nominal control of the corresponding LDR, i.e.,

$$E\{\underline{u}(t)\} = \underline{u}_d^*(t) .$$

where $\underline{u}(t)$ is the optimal nominal control of the LSR and $\underline{u}_d^*(t)$ is the deterministic optimal nominal control of the LDR. Taking the expected value of (2.26) results in

$$E\{\underline{u}(t)\} = -C(t)E\{\hat{\underline{x}}(t)\} ,$$

Using Property 2.1, $\quad = -C(t)\underline{x}_d^*(t)$, $\qquad$ (2.35)

Using (2.8), $\quad = \underline{u}_d^*(t)$.

Sensitivity Analysis of the Original LSR

The variables of a given LSR that are subject to modelling errors are the state coefficient matrix F; the control coefficient matrix G; the measurement matrix H; the initial value of the error covariance matrix, P_o; the initial value of the state $\underline{x}_o$; and the noise covariances Q_w and Q_v. In the following, the differential equations for the mean and mean square trajectory sensitivity are derived for the case in which the only modelling errors are in $F(\alpha,t)$, $G(\alpha,t)$, and $P_o(\alpha)$, where α is a scalar parameter; a more general case will be derived in Appendix A. Let $\underline{\zeta}(t)$ and $\hat{\underline{\zeta}}(t)$ be the trajectory sensitivity and the estimate sensitivity of the original (uncompensated) LSR, respectively, and be defined by

$$\underline{\zeta}(t) = \left.\frac{\partial \underline{x}}{\partial \alpha}\right|_{\alpha=\alpha_n} , \qquad (2.37)$$

and $$\hat{\underline{\zeta}}(t) = \left.\frac{\partial \hat{\underline{x}}}{\partial \zeta}\right|_{\alpha=\alpha_n} . \tag{2.38}$$

Using (2.13) and (2.20), it can be shown that

$$\begin{aligned} \dot{\underline{\zeta}} &= F(t)\underline{\zeta} - G(t)C(t)\hat{\underline{\zeta}} + F_\alpha(t)\underline{x}(t) \\ &\quad -[G_\alpha(t)C(t) + G(t)C_\alpha(t)]\hat{\underline{x}}(t),\ \underline{\zeta}(t_0)=\underline{0}, \end{aligned} \tag{2.39}$$

where

$$F_\alpha(t) \triangleq \left.\frac{\partial F(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n}, G_\alpha(t) \triangleq \left.\frac{\partial G(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n}, \text{ and}$$

$$C_\alpha(t) \triangleq \left.\frac{\partial C(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n},$$

and all the vectors and matrices are evaluated at the nominal parameter value, α_n. Similarly, after using (2.14) and (2.20) in (2.21), it can be shown that

$$\begin{aligned} \dot{\hat{\underline{\zeta}}} &= K(t)H(t)\underline{\zeta} + [F(t) - G(t)C(t) - K(t)H(t)]\hat{\underline{\zeta}} \\ &\quad +[F_\alpha(t)-G_\alpha(t)C(t)-G(t)C_\alpha(t)-K_\alpha(t)H(t)]\hat{\underline{x}}+K_\alpha(t)H(t)x(t) \\ &\quad + K_\alpha(t)\underline{v}(t),\ \hat{\underline{\zeta}}(t_0) = \underline{0}. \end{aligned} \tag{2.40}$$

C_α is given by (2.41) and (2.42) below:

$$C_\alpha = B^{-1}(t)[G_\alpha^T(t)S_2(t) + G^T(t)S_{2\alpha}(t)], \tag{2.41}$$

$$\begin{aligned} \dot{S}_{2\alpha} &= -S_{2\alpha}[F-GB^{-1}G^TS_2]-[F-GB^{-1}G^TS_2]^TS_{2\alpha} -[S_2F_\alpha+F_\alpha^TS_2] \\ &\quad + S_2[G_\alpha B^{-1}G^T + GB^{-1}G_\alpha^T]S_2,\quad S_{2\alpha}(t_f) = 0, \end{aligned} \tag{2.42}$$

where the subscript α denotes a partial differentiation with respect to α and evaluated at $\alpha = \alpha_n$. And $K_\alpha(t)$ is given by (2.43) and (2.44) below:

$$K_\alpha = P_\alpha(t)H^T(t)Q_v^{-1}(t) , \tag{2.43}$$

$$\dot{P}_\alpha = P_\alpha[F-PH^TQ_v^{-1}H]^T + [F-PHQ_v^{-1}H]P_\alpha + [F_\alpha P+PF_\alpha^T],$$
$$P\ (t_o) = \left.\frac{\partial P_o}{\partial \alpha}\right|_{\alpha=\alpha_n} , \tag{2.44}$$

where the subscript α has the same meaning as in the preceding. The desired differential equations can be expressed in a compact form in terms of an augmented state defined below.

$$\underline{\nu}(t) \triangleq \begin{bmatrix} \underline{\zeta}(t) \\ \hat{\underline{\zeta}}(t) \\ \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix} . \tag{2.45}$$

Equations (2.39), (2.40), (2.24) with $\underline{u}(t)$ being substituted for via (2.26), and (2.27) with $\underline{u}(t)$ and $\underline{z}(t)$ being substituted for via (2.26) and (2.25), become in the augmented space

$$\dot{\underline{\nu}} = \Sigma(t)\underline{\nu} + \underline{\gamma}(t), \tag{2.46}$$

where

$$\Sigma(t) \triangleq \begin{bmatrix} F(t) & -G(t)C(t) & F_\alpha(t) & -G(t)C_\alpha(t)-G_\alpha(t)C(t) \\ K(t)H(t) & F(t)-G(t)C(t)-K(t)H(t) & K_\alpha(t)H(t) & F_\alpha(t)-G(t)C_\alpha(t)-G_\alpha(t)C(t) \\ 0 & 0 & F(t) & -G(t)C(t) \\ 0 & 0 & K(t)H(t) & F(t)-G(t)C(t)-K(t)H(t) \end{bmatrix}$$

(2.47)

and $$\underline{\gamma}(t) \triangleq \begin{bmatrix} 0 \\ K_\alpha(t)\underline{v}(t) \\ \underline{w}(t) \\ K(t)\underline{v}(t) \end{bmatrix} . \quad (2.48)$$

Taking the expected value of both sides of (2.46), one has

$$\dot{\underline{\mu}}_\nu = \Sigma(t)\underline{\mu}_\nu,\ \underline{\mu}_\nu(t_o) = [\underline{0} \quad \underline{0} \quad x_o^T \quad \underline{x}_o^T]^T, \quad (2.49)$$

where $$\underline{\mu}_\nu \triangleq E\{\underline{\nu}(t)\}.$$

Equation (2.49) represents the differential equations for the mean trajectory sensitivity, the mean estimate sensitivity, the mean state and the mean estimate. In order to obtain the differential equations for the mean square trajectory sensitivity, the following definitions are introduced:

$$P_{xy}(t) \triangleq E\{\underline{x}(t)\underline{y}^T(t)\}, \quad (2.50)$$

and $$\underline{\mu}_x(t) \triangleq E\{\underline{x}(t)\} , \tag{2.51}$$

for any arbitrary $\underline{x}(t)$ and $\underline{y}(t)$.

Using the above definition,

$$P_{\nu\nu}(t) = E\{\underline{\nu}(t)\underline{\nu}^T(t)\} = \begin{bmatrix} P_{\zeta\zeta} & P_{\zeta\hat{\zeta}} & P_{\zeta x} & P_{\zeta\hat{x}} \\ P^T_{\zeta\hat{\zeta}} & P_{\hat{\zeta}\hat{\zeta}} & P_{\hat{\zeta}x} & P_{\hat{\zeta}\hat{x}} \\ P^T_{\zeta x} & P^T_{\hat{\zeta}x} & P_{xx} & P_{x\hat{x}} \\ P^T_{\zeta\hat{x}} & P^T_{\hat{\zeta}\hat{x}} & P^T_{x\hat{x}} & P_{\hat{x}\hat{x}} \end{bmatrix} \tag{2.52}$$

Differentiating both sides of (2.52) with respect to t, one has

$$\dot{P}_{\nu\nu}(t) = E\{\underline{\nu}(t)\dot{\underline{\nu}}^T(t) + \dot{\underline{\nu}}^T(t)\underline{\nu}(t)\} . \tag{2.53}$$

Using (2.46), equation (2.53) becomes

$$\dot{P}_{\nu\nu}(t) = \Sigma(t)P_{\nu\nu} + P_{\nu\nu}\Sigma^T(t) + \Omega, \; P_{\nu\nu}(t_o) = Q_o, \tag{2.54}$$

where, see Appendix A,

$$\Omega \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_\alpha(t)Q_v(t)K_\alpha^T(t) & 0 & K_\alpha(t)Q_v(t)K^T(t) \\ 0 & 0 & Q_w(t) & 0 \\ 0 & K(t)Q_v(t)K_\alpha^T(t) & 0 & K(t)Q_v(t)K^T(t) \end{bmatrix}, \tag{2.55}$$

and Q_o is known.

The solution $P_{\nu\nu}(t)$ of (2.54) gives the mean square trajectory sensitivity, $P_{\zeta\zeta}(t)$, of the original LSR.

Property 2.3:

The mean trajectory sensitivity and the mean estimate sensitivity of an optimal nominal LSR are equal, i.e.,

$$E\{\underline{\zeta}(t)\} = E\{\hat{\underline{\zeta}}(t)\} .$$

Proof:

Subtracting (2.39) from (2.40), one has

$$(\dot{\hat{\underline{\zeta}}}-\dot{\underline{\zeta}}) = [F(t)-K(t)H(t)](\hat{\underline{\zeta}}-\underline{\zeta}) + [F_{\alpha}(t)-K_{\zeta}(t)H(t)](\hat{\underline{x}}(t)-\underline{x}(t))$$
$$+ K_{\alpha}(t)v(t), \quad \hat{\zeta}(t_0)-\zeta(t_0) = \underline{0} .$$

Taking the expected value of the above equation, using (2.17) and (2.31), and assuming that the order of differentiation and taking the expectation can be interchanged, one has

$$\frac{d}{dt} E\{\hat{\underline{\zeta}}-\underline{\zeta}\} = [F(t)-K(t)H(t)]E\{\hat{\underline{\zeta}}-\underline{\zeta}\}, E\{\hat{\underline{\zeta}}(t_0)-\underline{\zeta}(t_0)\} = \underline{0}. \tag{2.56}$$

Hence, the solution of (2.56) is identically equal to zero, i.e.,

$$E\{\hat{\underline{\zeta}}(t) - \underline{\zeta}(t)\} = \underline{0} \quad \text{for all } t .$$

Hence, $$E\{\hat{\underline{\zeta}}(t)\} = E\{\underline{\zeta}(t)\} \quad \text{for all } t . \tag{2.57}$$

Or, using the definition given by (2.51),

$$\underline{\mu}_{\hat{\zeta}}(t) = \underline{\mu}_{\zeta}(t) \quad \text{for all } t. \tag{2.58}$$

Summary

The mathematical formulations of LDR's and LSR's have been given without proof because they are standard results and readily available in literature, for example, see [51-53]. Useful relationships between the optimal nominal control and trajectory of an LSR and those of the corresponding LDR have been established in Properties 2.1 and 2.2. The differential equations for the mean trajectory sensitivity and the mean square trajectory sensitivity due to modelling errors in F, G and P_0 (for a more general case, see Appendix A) are given by (2.49), and (2.54), respectively. It is a good design practice, before commencing a sensitivity compensation design, to calculate the mean square trajectory sensitivity due to various parameters of the original LSR, and only those parameters that give rise to significant mean square trajectory sensitivity should be compensated. The mean trajectory sensitivity given by (2.49) describes the average behaviour of the trajectory sensitivity. Property 2.3 establishes that the mean trajectory sensitivity and the mean estimate sensitivity of an optimal nominal LSR are equal.

CHAPTER III

TRAJECTORY SENSITIVITY COMPENSATION

This chapter presents the main body of the research. The problem statement is presented which describes how the problem of trajectory sensitivity occurs in an actual LSR and what should be done about it. The mathematical formulations of the compensation method of this research are given. Then the compensation method is applied to reduce the trajectory sensitivity of an LSR which is subject to general modelling errors in the state coefficient matrix $F(\alpha,t)$, the control coefficient matrix $G(\alpha,t)$, the measurement matrix $H(\alpha,t)$, the initial estimate error covariance matrix $P_o(\alpha)$, the initial state $\underline{x}_o(\alpha)$, and the noise covariances $Q_w(\alpha,t)$ and $Q_v(\alpha,t)$, where α is a scalar constant parameter. Details of the derivations of the necessary conditions for this problem are given in Appendix B. And finally an extension of the compensation method to a multi-parameter, multivariable LSR, and a summary are given, respectively.

The Problem Statement

It is a common practice in the design of an LSR to design the deterministic controller and the Kalman filter off-line on the basis of the plant nominal parameter value, $\underline{\alpha}_n$, because the actual value of the parameter $\underline{\alpha}$ is unknown in most cases. The deterministic controller gain C(t) is calculated from (2.10) and (2.11), and the Kalman gain K(t) from (2.28) and (2.29). In both cases $\underline{\alpha} = \underline{\alpha}_n$ is used. Then, the gains, C(t) and K(t), are connected to the actual plant with the actual parameter $\underline{\alpha}$ to form an actual LSR, as shown in Figure 4. The dynamical equations of the actual LSR are given by

$$\dot{\underline{x}} = F(\underline{\alpha},t)\underline{x} - G(\underline{\alpha},t)C(\underline{\alpha}_n,t)\hat{\underline{x}} + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o(\underline{\alpha}); \hat{\underline{x}} \tag{3.1}$$

$$\dot{\hat{\underline{x}}} = K(\underline{\alpha}_n,t)\underline{x} + [F(\underline{\alpha},t) - G(\underline{\alpha},t)C(\underline{\alpha}_n,t) - K(\underline{\alpha}_n,t)H(\underline{\alpha},t)]\,x$$

$$+ K(\underline{\alpha}_n,t)\underline{v}(t), \quad \hat{\underline{x}}(t_o) = \underline{x}_o(\underline{\alpha}), \tag{3.2}$$

where $\underline{\alpha}_n$ has been shown as arguments in C(t) and K(t) in order to emphasize the fact that they are calculated on the basis of $\underline{\alpha}_n$.

Suppose that the value of the parameter $\underline{\alpha}$ of the actual plant were exactly the same as the nominal value, i.e., $\underline{\alpha} = \underline{\alpha}_n$, one would have a nominal LSR in which the actual trajectory of the LSR would be exactly the same as that calculated during the design stage. The dynamical equations of the nominal LSR (or the LSR during the design stage) would be

$$\dot{\underline{x}} = F(\underline{\alpha}_n,t)\underline{x} - G(\underline{\alpha}_n,t)C(\underline{\alpha}_n,t)\hat{\underline{x}} + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o(\underline{\alpha}); \tag{3.3}$$

$$\dot{\hat{\underline{x}}} = K(\underline{\alpha}_n,t)\underline{x} + [F(\underline{\alpha}_n,t) - G(\underline{\alpha}_n,t)C(\underline{\alpha}_n,t) - K(\underline{\alpha}_n,t)H(\underline{\alpha}_n,t)]\,\hat{\underline{x}}$$
$$+ K(\underline{\alpha}_n,t)\underline{v}(t),\ \hat{\underline{x}}(t_o) = \underline{x}_o(\underline{\alpha}) \ . \qquad (3.4)$$

Let $\underline{x}(\underline{\alpha},t)$ be the trajectory of the actual LSR defined by (3.1) and (3.2), and $\underline{x}_n(\underline{\alpha}_n,t)$ be the trajectory of the nominal LSR defined by (3.3) and (3.4). It is obvious from (3.1) through (3.4) that the actual trajectory, $\underline{x}(\underline{\alpha},t)$ will deviate from the nominal trajectory, $\underline{x}_n(\underline{\alpha}_n,t)$ by an amount of $\Delta\underline{x}(\underline{\alpha},t)$ given by

$$\Delta\underline{x}(\underline{\alpha},t) = \underline{x}(\underline{\alpha},t) - \underline{x}_n(\underline{\alpha}_n,t) \qquad (3.5)$$

In some cases, $\Delta\underline{x}(\underline{\alpha},t)$ or the deviation of the actual trajectory from the nominal can become so large that the actual LSR is not suitable for the work it has been designed to do. In practical applications, the deviation of the actual trajectory from the nominal, $\Delta\underline{x}(\underline{\alpha},t)$, is inevitable because modelling errors are inevitable. Modelling errors can arise from manufacturing tolerances, measuring instrument errors, changes in the environment (e.g., changes in temperature or pressure cause some changes in the electrical resistance of some materials), aging of components, and so on. It is the main objective of this research to obtain a compensation method that ensures that $\Delta\underline{x}(\underline{\alpha},t)$ of an actual LSR will be small even though the LSR is operating under the influence of modelling errors.

The Compensation Method

The compensation method is based on the concept of trajectory sensitivity. It utilizes a two-degree-of-freedom control structure which enables a reduction in the trajectory sensitivity to be effected independently of the optimal nominal trajectory. This means that the desired trajectory sensitivity and the desired optimal trajectory can be specified independently. A two-degree-of-freedom control structure has been known and in use to effect a sensitivity reduction in deterministic control systems for some time [46]. Price and Deyst [36] and Grübel and Kreisselmeir [32] employ the structure to reduce the trajectory sensitivity of deterministic control systems. The concept of the sensitivity method with a two-degree-of-freedom control structure for deterministic systems is well-delineated by Crane and Stubberud in [49]; in fact, in this research, the first version of the development of an extension of the concept to stochastic systems is partially guided by their work. The approach of this version is heuristic in nature. The second version of the development is partially guided by Grübel and Kreisselmeir's work in [32].

The following is the first version of the development. In general, the solution of an LSR such as that defined by (2.13), (2.20), and (2.21) can be represented by

$$\dot{\underline{x}} = \underline{f}(\underline{x}',\underline{u}',\underline{\alpha},\underline{\omega},t),\ E\{\underline{x}'(t_0)\} = \underline{x}_0',\ t_0 \leq t \leq t_f \qquad (3.6)$$

where $\underline{x}'$ is an augmented state consisting of the state and the estimate, $\underline{u}'$ is a control, $\underline{\alpha}$ is a parameter vector and $\underline{\omega}$ are sample functions of a vector stochastic process. Let $\underline{\alpha}_n$ be the nominal value of $\underline{\alpha}$, $\underline{x}_n'(t) \triangleq \underline{x}'(\underline{\alpha}_n,t)$ be the nominal trajectory, and $\underline{u}_n'(t)$ be the corresponding nominal control. Then, one has a nominal system given by

$$\dot{\underline{x}}_n' = \underline{f}(\underline{x}_n',\underline{u}_n',\underline{\alpha}_n,\underline{\omega},t),\ E\{\underline{x}_n'(t_0)\} = \underline{x}_0',\ t_0 \leq t \leq t_f \qquad (3.7)$$

The system given by (3.6) as is does not have any built-in protection against the trajectory sensitivity problem caused by the inevitable parameter mismatches. Therefore, it is the purpose of this research to incorporate into the system some type of sensitivity compensation that will reduce the system trajectory sensitivity to an insignificant level. To do this, consider a modification of the original control structure of $\underline{f}(\cdot)$ to be of the form

$$\underline{u}(\underline{x}',t) = \underline{u}_1'(t) - s_1(t)\underline{x}'(\underline{\alpha},t), \qquad (3.8)$$

where $\underline{u}_1'(t)$ and $S_1(t)$ are a vector and a matrix, respectively, and $\underline{x}'(\underline{\alpha},t)$ is the actual state. For convenience of reference, refer to the modified system by $\underline{g}(\underline{x}',\underline{\alpha},\underline{\omega},t)$ or mathematically, the new system is

$$\underline{\dot{x}}' = \underline{g}(\underline{x}',\underline{\alpha},\underline{\omega},t), \quad t_0 \le t \le t_f , \tag{3.9}$$

where $\underline{g}(\cdot)$ is such that

$$\underline{g}(\underline{x}',\underline{\alpha},\underline{\omega},t) = \underline{f}(\underline{x}',\underline{u}'(\underline{x}',t),\underline{\alpha},\underline{\omega},t), \tag{3.10}$$

where $\underline{u}'(\underline{x}',t)$ is given by (3.8).

Before proceeding any further, the following assumptions are made:

(a_1) The parameter α is a deterministic scalar constant. An extension to the case where $\underline{\alpha}$ is a vector parameter will be given later;

(a_2) The parameter error, $\delta\alpha = \alpha - \alpha_n$, is sufficiently small such that only the first order terms are required to describe the system behaviour relative to the nominal; and

(a_3) $\underline{u}_1'(t)$ in (3.8) is chosen such that

$$g(\underline{x}',\alpha_n,\underline{\omega},t) = \underline{f}(\underline{x}',\underline{u}_n',\alpha_n,\underline{\omega},t), \tag{3.11}$$

i.e., the system in (3.9) reduces to the nominal system in (3.7) when there are no parameter errors. This is known as the "nominal equivalence condition."

In order to reduce the trajectory sensitivity, $\underline{g}(\cdot)$ is chosen such that the solution of (3.9), $\underline{x}'(\alpha,t)$, minimizes some functions of (a) the trajectory error

$$\Delta\underline{x}'(\alpha,t) = \underline{x}'(\alpha,t) - \underline{x}_n'(\underline{\alpha}_n,t), \tag{3.12}$$

and (b) the rate error

$$\Delta \underline{g}(\underline{x}',\alpha,\underline{\omega},t) = \underline{g}(\underline{x}',\alpha,\underline{\omega},t) - \underline{f}(\underline{x}',\underline{u}_n',\alpha,\underline{\omega},t). \qquad (3.13)$$

The trajectory and the rate errors can be combined into a quadratic cost function

$$J_s = E\{ \|\Delta \underline{x}'(\alpha,t_f)\|^2_{Q_f} + \int_{t_o}^{t_f} [\|\Delta \underline{x}'(\alpha,t)\|^2_{Q_1} + \|\Delta(\underline{x}',\alpha,\omega,t)\|^2_{Q_3}]dt\} \qquad (3.14)$$

where Q_f and Q_1 are $n \times n$ symmetric positive semi-definite matrices and Q_3 is a symmetric positive definite matrix. Next, J_s in (3.14) will be equivalently expressed in terms of the trajectory sensitivity instead of $\Delta \underline{x}'(\cdot)$ and $\Delta \underline{g}(\cdot)$. Using assumptions ($a_1$) and ($a_2$), the trajectory error from (3.12) is given by

$$\begin{aligned} \Delta \underline{x}'(\alpha,t) &= \underline{x}'(\alpha,t) - \underline{x}_n'(\alpha_n,t), \\ &\doteq \left.\frac{\partial \underline{x}'(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n} (\alpha-\alpha_n) \\ &= \underline{\xi}'(t)\ \delta\alpha, \end{aligned} \qquad (3.15)$$

where $\underline{\xi}'(t) \triangleq \left.\frac{\partial \underline{x}'(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n}$ represents the first-order trajectory sensitivity. Also, using assumptions (a_1) and (a_2), one has the following

$$\underline{f}(\underline{x}',\underline{u}_n',\alpha,\underline{\omega},t) = \underline{f}(\underline{x}_n',\underline{u}_n',\alpha_n,\underline{\omega},t) + \left[\frac{\partial \underline{f}}{\partial \underline{x}'} \cdot \frac{\partial \underline{x}'}{\partial \alpha} + \frac{\partial \underline{f}}{\partial \alpha}\right]_{\alpha=\alpha_n} \delta\alpha \quad (3.16)$$

$$\underline{g}(\underline{x}',\alpha,\underline{\omega},t) = \underline{g}(\underline{x}_n',\alpha_n,\underline{\omega},t) + \left[\frac{\partial \underline{g}}{\partial \underline{x}'} \cdot \frac{\partial \underline{x}'}{\partial \alpha} + \frac{\partial \underline{g}}{\partial \alpha}\right]_{\alpha=\alpha_n} \delta\alpha. \quad (3.17)$$

Hence, using (3.16) and (3.17), $\Delta\underline{g}(\cdot)$ from (3.13) becomes

$$\Delta\underline{g}(\cdot) = \underline{g}(\underline{x}_n',\alpha_n,\underline{\omega},t) - \underline{f}(\underline{x}_n',\underline{u}_n,\alpha_n,\underline{\omega},t) + \left[\frac{\partial \underline{g}}{\partial \underline{x}'} \cdot \frac{\partial \underline{x}'}{\partial \alpha} + \frac{\partial \underline{g}}{\partial \alpha} - \frac{\partial \underline{f}}{\partial \underline{x}'} \cdot \frac{\partial \underline{x}'}{\partial \alpha} - \frac{\partial \underline{f}}{\partial \alpha}\right]_{\alpha=\alpha_n} \delta\alpha. \quad (3.18)$$

Next, $\underline{u}_1'(t)$ of (3.8) is to be chosen such that the nominal equivalence condition in (3.11) is satisfied, i.e.,

$$\underline{u}'(\underline{x}',t) = \underline{u}_n'(t) = \underline{u}_1'(t) - S_1(t)\underline{x}'(\underline{\alpha}_n,t). \quad (3.19)$$

Hence, $$\underline{u}_1'(t) = \underline{u}_n'(t) + S_1(t)\underline{x}'(\alpha_n,t). \quad (3.20)$$

Substituting $\underline{u}_1'(t)$ from (3.20) in (3.8), the control $\underline{u}'(\underline{x},t)$ is given by

$$\underline{u}'(\underline{x}',t) = \underline{u}_n'(t) + S_1(t)[\underline{x}'(\alpha_n,t) - \underline{x}'(\alpha,t)]. \quad (3.21)$$

Therefore, using the control $\underline{u}'(\underline{x},t)$ of (3.21), the nominal equivalence condition of (3.11) is satisfied, and consequently (3.18) becomes

$$\Delta\underline{g}(\underline{x}',\alpha,\underline{\omega},t) = \left(\frac{\partial \underline{g}}{\partial \underline{x}'} - \frac{\partial \underline{f}}{\partial \underline{x}'}\right) \frac{\partial \underline{x}'}{\partial \alpha}\bigg|_{\alpha=\alpha_n} \delta\alpha + \left(\frac{\partial \underline{g}}{\partial \alpha} - \frac{\partial \underline{f}}{\partial \alpha}\right)_{\alpha=\alpha_n} \delta\alpha \quad (3.22)$$

From (3.10), one has

$$\frac{\partial \underline{g}}{\partial \underline{x}'} = \frac{\partial \underline{f}}{\partial \underline{x}'} + \frac{\partial \underline{f}}{\partial \underline{u}'} \cdot \frac{\partial u'}{\partial x'} , \tag{3.23}$$

and

$$\frac{\partial \underline{g}}{\partial \alpha} = \frac{\partial \underline{f}}{\partial \alpha} . \tag{3.24}$$

Using (3.23) and (3.24) in (3.22), one has

$$\Delta \underline{g}(\underline{x}',\alpha,\underline{\omega},t) = \frac{\partial f}{\partial u'} \cdot \frac{\partial \underline{u}'}{\partial \underline{x}'} \cdot \frac{\partial \underline{x}'}{\partial \alpha}\bigg|_{\alpha=\alpha_n} \delta\alpha$$

which yields, using (3.21),

$$\Delta \underline{g}(\underline{x}',\alpha,\underline{\omega},t) = - \frac{\partial \underline{f}(\underline{x}',\underline{u}',\alpha,\underline{\omega},t)}{\partial \underline{u}'}\bigg|_{\alpha=\alpha_n} S_1(t)\cdot\underline{\xi}'(t)\cdot\delta\alpha \tag{3.25}$$

Using (3.15) and (3.25) in (3.14), the expression for J_s becomes

$$J_s = (\delta\alpha)^2 \cdot E\{||\underline{\xi}'(t_f)||^2_{Q_f} + \int_{t_o}^{t_f} [||\underline{\xi}'(t)||^2_{Q_1} + ||S_1(t)\underline{\xi}'(t)||^2_{Q_2}]dt\} \tag{3.26}$$

where $Q_2 \triangleq \frac{\partial \underline{f}^T}{\partial \underline{u}'} \cdot Q_3 \cdot \frac{\partial \underline{f}}{\partial \underline{u}'}\bigg|_{\alpha=\alpha_n}$.

Then, since the parameter error $\delta\alpha$ is uncontrollable and the matrix Q_2 is positive definite, it is sufficient to minimize

$$\min_{S_1(t)} J_s = \min_{S(t)} E\{\|\underline{\xi}'(t_f)\|^2_{Q_f} + \int_{t_o}^{t_f} [\|\underline{\xi}'(t)\|^2_{Q_1} + \|S_1(t)\underline{\xi}'(t)\|^2_{Q_2}]dt\} \tag{3.27}$$

Next, the differential equations for the trajectory sensitivity, $\underline{\xi}'(t)$, will be obtained. From (3.9), (3.10), and assumption (a_1), one has

$$\dot{\underline{x}}' = f(\underline{x}',\underline{u}'(\underline{x}',t),\alpha,\underline{\omega},t) \tag{3.28}$$

By partially differentiating (3.28) with respect to ξ and using (3.21), one has the desired sensitivity equation:

$$\dot{\underline{\xi}}' = \left.\frac{\partial \underline{f}}{\partial \underline{x}'}\right|_{\alpha=\alpha_n} \underline{\xi}'(t) - \left.\frac{\partial \underline{f}}{\partial \underline{u}'}\right|_{\alpha=\alpha_n} S_1(t)\cdot\underline{\xi}'(t) + \left.\frac{\partial \underline{f}}{\partial \alpha}\right|_{\alpha=\alpha_n} \tag{3.29}$$

In summary, in order to minimize the trajectory sensitivity of the system of (3.6), the control structure of the system is modified as given by (3.8) where $\underline{u}_1'(t)$ is chosen such that the nominal equivalence condition of (3.11) is satisfied and $S_1(t)$ is chosen such that the sensitivity performance index, J_s, is minimized subject to the constraints set by (3.29). However, an inspection of (3.21) reveals that the control $\underline{u}'(\underline{x}',t)$ is not practical; because it must be generated from $\underline{u}_n'(t)$ and $\underline{x}'(\alpha_n,t)$. In practice, it is not possible to generate $\underline{u}_n'(t)$ and $\underline{x}'(\alpha_n,t)$, see (3.6); because the actual time evolution of the disturbance $\underline{\omega}(t)$

(plant and measurement noises) is unknown and cannot be isolated from the physical system. This means that in practice the nominal equivalence condition in (3.11) cannot be satisfied for every $\underline{\omega}(t)$. In this research, the expected values of $\underline{u}_n'(t)$ and $\underline{x}'(\alpha_n,t)$, i.e. $E\{\underline{u}_n'(t)\}$ and $E\{\underline{x}'(\alpha_n,t)\}$, are used instead of $\underline{u}_n'(t)$ and $\underline{x}'(\alpha_n,t)$; and $\underline{u}'(\underline{x}',t)$ now becomes

$$\underline{u}'(\underline{x}',t) = E\{\underline{u}_n'(t)\} + S_1(t)[E\{\underline{x}'(\alpha_n,t)\} - \underline{x}'(\alpha,t)]. \qquad (3.30)$$

The reasons for this choice are because (a) they are known and can be generated in practice; and (b) the nominal equivalence condition is satisfied on the average. This means that the compensated system will reduce on the average to the original (uncompensated) system when the modelling errors are zero.

The following is the second version of the development. Instead of going through all the lengthy arguments above to arrive at the results given by (3.27) and (3.30), an equally valid but simpler argument is given in the following. First, the feedback control $\underline{u}'(\underline{x}',t)$ is chosen according to (3.30) without any concern to the nominal equivalence condition at present. Second, $S_1(t)$ is chosen such that the mean square deviation of the actual trajectory from the nominal is minimized, i.e.,

$$\min_{S_1(t)} J_1 = \min_{S_1(t)} E\{ \|\Delta\underline{x}'(t_f)\|^2_{Q_f} + \int_{t_0}^{t_f} \|\Delta\underline{x}'(t)\|^2_{Q_1} dt\}, \tag{3.31}$$

where $\Delta\underline{x}'(\cdot)$ is the same as in (3.12). Similarly, it can be shown that (3.31) is equivalent to

$$\min_{S_1(t)} J_1 = \min_{S_1(t)} E\{ \|\underline{\xi}'(t_f)\|^2_{Q_f} + \int_{t_0}^{t_f} \|\underline{\xi}'(t)\|^2_{Q_1} dt\}. \tag{3.32}$$

The optimization problem defined by (3.32), as it stands now, may not be meaningful because the elements of matrix $S_1(t)$ may become infinite. To ensure that $S_1(t)$ will remain finite, the integrand of (3.32) must be modified as follows. Define a vector,

$$\underline{\rho}'(t) = S_1(t)\underline{\xi}'(t), \tag{3.33}$$

and modify J_1 in (3.32) (denoted by J_2) as shown below.

$$\min_{S_1(t)} J_2 = \min_{S_1(t)} E\{ \|\underline{\xi}'(t_f)\|^2_{Q_f} + \int_{t_0}^{t_f} [\|\underline{\xi}'(t)\|^2_{Q_1} + \|\underline{\rho}'(t)\|^2_{Q_5}]dt\} , \tag{3.34}$$

where Q_5 is a symmetric positive-definite matrix. Or, equivalently,

$$\min_{S_1(t)} J_2 = \min_{S_1(t)} E\{ \|\underline{\xi}'(t_f)\|^2_{Q_f} + \int_{t_0}^{t_f} [\|\underline{\xi}'(t)\|^2_{Q_1} + \|S_1(t)\underline{\xi}'(t)\|^2_{Q_5}]dt\}. \tag{3.35}$$

The optimization defined by (3.35) and that by (3.27) are equivalent because the weight matrices Q_2 and Q_5 are arbitrary in design practice. Finally, the use of $E\{\underline{u}_n'(t)\}$ and $E\{\underline{x}'(\alpha_n,t)\}$ in (3.30) is justified by proving that on the average the compensated system reduces to the original system when the modelling errors are zero (see Property 3.3 to be presented).

The argument presented in the second version appears to be simpler than that in the first one. However, the argument in the first has merit in that it links the second term of the integrand in (3.27) to the rate error. This gives a little more insight into the physics of the problem even though the rate error is not of prime importance to this research.

Application of the Compensation Method to Reduce the Trajectory Sensitivity of an LSR

Application of the compensation method to reduce the trajectory sensitivity of an LSR with general modelling errors in $F(\alpha,t)$, $G(\alpha,t)$,$H(\alpha,t)$, $P_0(\alpha)$,$\underline{x}_0(\alpha)$,$Q_W(\alpha,t)$ and $Q_V(\alpha,t)$, where α is a scalar parameter, is to be considered. Extension of the compensation method to a multiparameter case is given later. Consider the LSR problem defined as follows. Find a control $\underline{u}(t)$ such that the performance index

$$J = E\{\frac{1}{2} \|\underline{x}(t_f)\|^2_{S_f} + \frac{1}{2}\int_{t_0}^{t_f} [\|\underline{x}(t)\|^2_{A(t)} + \|\underline{u}(t)\|^2_{B(t)}]dt\} \quad (3.36)$$

is minimized subject to

$$\dot{\underline{x}} = F(\alpha,t)\underline{x} + G(\alpha,t)\underline{u}(t) + \underline{w}(t), E\{\underline{x}(t_0) = \underline{x}_0(\alpha),\ \mathrm{var}\{\underline{x}(t_0)\} = P_0(\alpha)\ , \quad (3.37)$$

and

$$\underline{z} = H(\alpha,t)\underline{x} + \underline{v}(t)\ . \quad (3.38)$$

where $\underline{x}(t)$ is an n-vector, $\underline{u}(t)$ is an r-vector, S_f and $A(t)$ are symmetric positive semi-definite matrices, $B(t)$ is a symmetric positive definite matrix, $\underline{z}(t)$ is an l-measurement vector, $\underline{w}(t)$ and $\underline{v}(t)$ are the process and measurement white Gaussian noises, respectively, with the following properties:

$$E\{\underline{w}(t)\} = \underline{0}\ , \quad (3.39)$$

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = Q_w(\alpha,t)\delta(t-\tau), \quad (3.40)$$

$$E\{\underline{v}(t)\} = \underline{0}, \quad (3.41)$$

$$E\{\underline{v}(t)\underline{v}^T(\tau)\} = Q_v(\alpha,t)\delta(t-\tau), \quad (3.42)$$

$$E\{\underline{v}(t)\underline{w}^T(\tau)\} - \underline{0}\ . \quad (3.43)$$

F, G, H, P_0, $\underline{x}_0$, Q_w and Q_v are dependent on the scalar parameter α. The solution to this LSR problem is given by

$$\dot{\underline{x}} = F(\alpha,t)\underline{x} + G(\alpha,t)\underline{u}(\alpha,t) + \underline{w}(t),\ E\{\underline{x}(t_0)\} = \underline{x}_0(\alpha); \quad (3.44)$$

$$\dot{\hat{\underline{x}}} = F(\alpha,t)\hat{\underline{x}} + G(\alpha,t)\underline{u}(\alpha,t) + K(\alpha,t)[H(\alpha,t)\underline{x} + \underline{v}(t) - H(\alpha,t)\hat{\underline{x}}],\ \hat{\underline{x}}(t_0) = \underline{x}_0(\alpha); \quad (3.45)$$

$$\underline{u}(t) = -C(\alpha,t)\hat{\underline{x}}; \quad (3.46)$$

$$K(\alpha,t) = P(\alpha,t)H^T(t)Q_v^{-1}(\alpha,t); \quad (3.47)$$

$$\dot{P}(\alpha,t) = F(\alpha,t)P(\alpha,t) + P(\alpha,t)F^T(\alpha,t) - K(\alpha,t)Q_v(t)K^T(\alpha,t) + Q_w(\alpha,t),\ P(t_0) = P_0(\alpha); \quad (3.48)$$

$$C(\alpha,t) = B^{-1}(t)G^T(\alpha,t)S_2(\alpha,t); \quad (3.49)$$

$$\dot{S}_2(\alpha,t) = -S_2(\alpha,t)F(\alpha,t) - F^T(\alpha,t)S_2(\alpha,t) + C^T(\alpha,t)B(t)C(\alpha,t) - A(t),\ S_2(t_f) = S_f, \quad (3.50)$$

where $\hat{x}$ is the state estimate given by the Kalman filter, $K(t)$ and $C(t)$ are the Kalman filter gain and the deterministic controller gain, respectively. Equations (3.44) - (3.50) define the original (uncompensated) LSR which is assumed to be stable under the nominal conditions of zero modelling errors. In the following, the compensation method described above will be applied to this LSR to make its trajectory less sensitive to the modelling errors. Guided by (3.30), the modified control structure is chosen as

$$\underline{u}[\hat{\underline{x}}(\alpha,t),t] = E\{\underline{u}(\alpha_n,t)\} + S(t)[E\{\underline{x}(\alpha_n,t)\} - \hat{\underline{x}}(\alpha,t)] \quad (3.51)$$

where $\underline{u}(\alpha_n,t)$ and $\underline{x}(\alpha_n,t)$ are the optimal nominal control and trajectory the LSR, and $\hat{\underline{x}}(\alpha,t)$ is the actual estimate of the actual state. Using Properties 2.1 *and* 2.2, (3.51) becomes

$$\underline{u}[\hat{\underline{x}}(\alpha,t),t] = \underline{u}_d^*(t) + S(t)[\underline{x}_d^*(t) - \hat{\underline{x}}(\alpha,t)], \quad (3.52)$$

where $\underline{u}_d^*(t)$ and $\underline{x}_d^*(t)$ are the optimal nominal control and trajectory of the corresponding LDR. Substituting $\underline{u}[\cdot]$ from (3.52) in (3.44) and (3.45) and rearranging results in the compensated system defined by (3.53) and (3.54) as shown below.

$$\begin{aligned}\dot{\underline{x}} = {} & F(\alpha,t)\underline{x} - G(\alpha,t)S(t)\hat{\underline{x}} + G(\alpha,t)[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)] \\ & + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o(\alpha); \quad (3.53)\end{aligned}$$

$$\begin{aligned}\dot{\hat{\underline{x}}} = {} & K(\alpha,t)H(\alpha,t)\underline{x} + [F(\alpha,t) - K(\alpha,t)H(\alpha,t) - G(\alpha,t)S(t)]\hat{\underline{x}} \\ & + K(\alpha,t)\underline{v}(t) + G(\alpha,t)[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)], \quad \hat{\underline{x}}(t_o) = \underline{x}_o(\alpha). \quad (3.54)\end{aligned}$$

It should be noted that the deterministic controller gain, C(t), has been eliminated from the compensated system; it is replaced by the compensating feedback gain, S(t). In the original (uncompensated) LSR, C(t) is computed off-line independently of the Kalman filter gain, K(t); but for the compensated system the compensating feedback gain $S(t)$ is calculated off-line as an integral part of the whole system, taking into account of the statistical nature of the trajectory sensitivity in the minimization of J_s. The control structure as chosen in (3.52) is not unique in that other structures are possible; but the structure in (3.52) is advantageous in that $\underline{u}_d^*(t)$ and $\underline{x}_d^*(t)$ are readily available from (2.8) and (2.9), and that it causes the compensated system to reduce on the average to the original LSR when the modelling errors are zero.

According to the compensation method, $S(t)$ must be chosen such that

$$J_s = E\{||\underline{\xi}(t_f)||^2_{Q_f} + \int_{t_o}^{t_f} [||\underline{\xi}(t)||^2_{Q_1} + ||S(t)\xi(t)||^2_{Q_2}]dt\} \quad (3.55)$$

is minimized subject to

$$\begin{aligned}\dot{\underline{\xi}} = {} & F(t)\underline{\xi} - G(t)S(t)\hat{\underline{\xi}} + F_\alpha(t)\underline{x}(t) - G_\alpha(t)S(t)\hat{\underline{x}}(t) \\ & + G_\alpha(t)[\quad \underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)], \quad \underline{\xi}(t_o) = \underline{\xi}_o ; \qquad (3.56)\end{aligned}$$

$$\begin{aligned}\dot{\hat{\underline{\xi}}} = {} & K(t)H(t)\underline{\xi} + [F(t) - K(t)H(t) - G(t)S(t)]\hat{\underline{\xi}} \\ & +[K_\alpha(t)H(t) + K(t)H_\alpha(t)]\underline{x} + [F_\alpha(t) - K_\alpha(t)H(t) \\ & - K(t)H_\alpha(t) - G_\alpha(t)S(t)]\hat{\underline{x}}(t) + G_\alpha(t)[\underline{u}_d^*(t) \\ & + S(t)\underline{x}_d^*(t)] + K_\alpha(t)\underline{v}(t), \quad \hat{\underline{\xi}}(t_o) = \underline{\xi}_o; \qquad (3.57)\end{aligned}$$

$$\dot{\underline{x}} = F(t)\underline{x} - G(t)S(t)\hat{\underline{x}} + G(t)[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)] + \underline{w}(t),$$
$$E\{\underline{x}(t_0)\} = \underline{x}_0; \tag{3.58}$$

$$\dot{\hat{\underline{x}}} = K(t)H(t)\underline{x} + [F(t) - K(t)H(t) - G(t)S(t)]\hat{\underline{x}} + K(t)\underline{v}(t)$$
$$+ G(t)[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)],\ \hat{\underline{x}}(t_0) = \underline{x}_0; \tag{3.59}$$

where $\underline{\xi} \triangleq \left.\dfrac{\partial \underline{x}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n}$ and $\hat{\underline{\xi}} \triangleq \left.\dfrac{\partial \hat{\underline{x}}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n}$

are the trajectory sensitivity and the estimate sensitivity, respectively, and the subscript α denotes a partial derivative with respect to α and evaluated at $\alpha = \alpha_n$. Equations (3.56) and (3.57) are obtained by partially differentiating (3.53) and (3.54) with respect to α and evaluating the results at $\alpha = \alpha_n$, respectively. Equations (3.55) - (3.59) represent an auxiliary optimization problem which yields S(t) as the optimum compensating feedback gain in the sense of a minimum trajectory sensitivity. The derivations of the necessary conditions that S(t) must satisfy are given in detail in Appendix B, and will not be repeated here. Only the results are given in the following. The necessary conditions are given by (B.42) - (B.47) and (B.61) and given below for reading continuity.

$$\dot{P}_{\theta\theta} = DP_{\theta\theta} + P_{\theta\theta}D^T + \Gamma,\quad P_{\theta\theta}(t_0) = Q_0; \tag{3.60}$$

$$\dot{\underline{\mu}}_\theta = D\underline{\mu} + \overline{\underline{\beta}},\quad \underline{\mu}_\theta(t_0) = \underline{\mu}_0\ ; \tag{3.61}$$

$$\dot{\Lambda} = -D^T\Lambda - \Lambda D - Q - R, \quad \Lambda(t_f) = S_f; \tag{3.62}$$

$$\underline{\dot{\lambda}} = -D^T\underline{\lambda} - 2\Lambda\underline{\bar{\beta}}, \quad \underline{\lambda}(t_f) = 0; \tag{3.63}$$

$$\begin{aligned}
&2Q_2SP_{\xi\xi} - 2G^T[(\Lambda_{\xi\xi} + \lambda^T_{\xi\hat{\xi}})P_{\xi\hat{\xi}} + (\Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})P_{\hat{\xi}\hat{\xi}} + (\Lambda_{\xi x} + \Lambda_{\hat{\xi}x})P^T_{\hat{\xi}x} \\
&\quad + (\Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})P^T_{\hat{\xi}\hat{x}} + (\Lambda^T_{\xi x} + \Lambda^T_{\xi\hat{x}})(P_{\xi\hat{x}} - \underline{\mu}_\xi\underline{x}^{*T}_d) \\
&\quad + (\Lambda^T_{\hat{\xi}x} + \Lambda^T_{\hat{\xi}\hat{x}})(P_{\hat{\xi}\hat{x}} - \underline{\mu}_\xi\underline{x}^{*T}_d) + (\Lambda_{xx} + \Lambda^T_{x\hat{x}})(P_{x\hat{x}} - \underline{x}^*_d\underline{x}^{*T}_d) \\
&\quad + (\Lambda_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}})(P_{\hat{x}\hat{x}} - \underline{x}^*_d\underline{x}^{*T}_d)] - 2G^T_\alpha[(\Lambda_{\xi\xi} + \Lambda^T_{\xi\hat{\xi}})(P_{\xi\hat{x}} - \underline{\mu}^*_\xi\underline{x}^{*T}_d) \\
&\quad + (\Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})(P_{\hat{\xi}\hat{x}} - \underline{\mu}_\xi\underline{x}^{*T}_d) + (\Lambda_{\xi x} + \Lambda_{\hat{\xi}x})(P_{x\hat{x}} - \underline{x}^*_d\underline{x}^{*T}_d) \\
&\quad + (\Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})(P_{\hat{x}\hat{x}} - \underline{x}^*_d\underline{x}^{*T}_d)] - G^T[\underline{\mu}_\xi(\underline{\lambda}^T_\xi + \underline{\lambda}^T_{\hat{\xi}}) + \underline{x}^*_d(\underline{\lambda}^T_x + \underline{\lambda}^T_{\hat{x}}) \\
&\quad - (\underline{\lambda}_x + \underline{\lambda}_{\hat{x}})\underline{x}^{*T}_d] - G^T_\alpha[\underline{x}^*_d(\underline{\lambda}^T_\xi + \underline{\lambda}^T_{\hat{\xi}}) - (\underline{\lambda}_\xi + \underline{\lambda}_{\hat{\xi}})\underline{x}^{*T}_d] = 0.
\end{aligned} \tag{3.64}$$

Equations (3.60) - (3.64) represent the necessary conditions in their final forms that must be satisfied. They also represent a two-point boundary-value problem (TPBVP) that must be solved. The name TPBVP derives from the fact the boundary conditions for $P_{\theta\theta}(t)$ and $\underline{\mu}_\theta(t)$ are known at the initial time, t_0, while those for $\Lambda(t)$ and $\underline{\lambda}(t)$ are known only at the final time, t_f. The TPBVP may be solved by any of the several available techniques such as the gradient algorithm, the quasi-linearization method, the invariant imbedding method, and so on.

In this research the steepest-descent gradient algorithm will be used to solve the TPBVP because of its simplicity and its important advantage in decoupling the state equations (3.60) and (3.61) from the costate equations (3.62) and (3.63). The decoupling property allows (3.60) and (3.61) to be solved independently of (3.62) and (3.63), and vice versa; thereby keeping the dimensions of the problem on hand small at any given instant. The steps involved in using the steepest-descent algorithm to numerically solve the TPBVP are as follows. Assuming that the interval $[t_0, t_f]$ is divided into s equal sub-intervals for the purpose of numerical integration of the D.E.'s. The discrete time instants are denoted by $t_0, t_1, t_2, \ldots, t_s$.

(a) Guess $S^i(t)$, for $t = t_0, t_1, t_2, \ldots, t_s$. The superscript i denotes the $i^{\underline{th}}$ iteration.

(b) Using $S^i(t)$, equations (3.60) and (3.61) are numerically integrated forward in time to solve for $P^i_{\theta\theta}(t)$ and $\underline{\mu}^i_\theta(t)$, $t = t_0, t_1, \ldots, t_s$.

(c) Using $S^i(t)$, equations (3.62) and (3.63) are numerically integrated backward in time for $\Lambda^i(t)$ and $\underline{\lambda}^i(t)$, $t = t_0, t_1, \ldots, t_s$.

(d) Using $S^i(t)$, $P^i_{\theta\theta}(t)$, $\underline{\mu}_\theta(t)$, $\Lambda^i(t)$ and $\underline{\lambda}^i(t)$ in (3.64) to calculate $(\frac{\partial\hat{H}}{\partial S}(t))^i$ for $t = t_0, t_1, \ldots, t_s$. Theoretically, $(\frac{\partial\hat{H}(t)}{\partial S})^i = 0$ for all t when S^i is optimal.

(e) Obtain an updated control $S^{i+1}(t)$ from

$$S^{i+1}(t) = S^{i}(t) - k(\frac{\partial \hat{H}}{\partial S})^{i}, \quad t = t_0, t_1, \ldots, t_s$$

where k is a positive number whose magnitude must be appropriately chosen.

(f) Repeat steps (b)-(e) with the superscript i replaced by i+1 until the optimum is approximately reached. When the optimum is reached the value of J_s will be at a minimum and $(\frac{\partial \hat{H}(t)}{\partial S})^{i} = 0$ for $t = t_0, t_1, \ldots, t_s$.

Comments concerning the selection of k in step (e) are proper. If k is too large, the divergence of the algorithm may occur. On the other hand, if k is too small the algorithm may be made unduly inefficient because of too slow a convergence rate. So far there has been no method that will analytically determine what the value of k should be in order to obtain an optimum convergence rate. Therefore, k must be chosen on a trial-and-error basis and then systematically varied. During the first few iterations, k should be made fairly small and progressively made larger during the subsequent iterations. The reason for doing this is as follows. During the first few iterations the initial guessed value of $S^{i}(t)$ may be far off the optimal value, which causes $(\frac{\partial \hat{H}}{\partial S})^{i}$ to be large in amplitude which in turn may cause $S^{i+1}(t)$ in step (e) to be over-corrected. After the first few iterations with small values of k, $S^{i}(t)$ is closer to the optimal value, and this in turn results in a smaller $(\frac{\partial \hat{H}}{\partial S})^i$; thereby k can be made larger to speed up the convergence rate without a high risk of over-correcting $S^i(t)$. The number of differential equations involved in this TPBVP is as follows. Since both $P_{\theta\theta}(t)$ and $\Lambda(t)$ are symmetric matrices, only their upper triangular elements need be considered. Therefore, the number of first-order differential equations for $P_{\theta\theta}(t)$ that must be solved is $2n(4n+1)$. Similarly, $2n(4n+1)$ for $\Lambda(t)$. Since matrix Γ in the R.H.S. of (3.60) is a function of $\underline{\mu}_\theta(t)$, (3.61) must also be solved; this involves another 4n differential equations. Equation (3.63) involving 4n equations must also be solved. Therefore, the number of equations is $4n(4n+3)$ in total. It will be shown that the number of equations can be reduced by 2n equations by using the relationships to be developed in the following.

Property 3.1: The average values of the state and the estimate of the compensated system are equal when modelling errors are zero, i.e., when $\alpha = \alpha_n$.

Proof: With $\alpha = \alpha_n$, subtracting (3.53) from (3.54) and rearranging, one has

$$(\dot{\hat{\underline{x}}} - \dot{\underline{x}}) = [F - KH](\hat{\underline{x}} - \underline{x}) + K\underline{v} - \underline{w}. \qquad (3.65)$$

Taking the expected value of (3.65) and assuming that the order of taking the expectation and the differentiation can be interchanged, one has

$$\frac{d}{dt} E\{\underline{x} - \underline{x}\} = [F - KH]E\{\hat{\underline{x}} - \underline{x}\} . \tag{3.66}$$

Since $\qquad E\{\hat{\underline{x}}(t_o) - \underline{x}(t_o)\} = 0$,

the solution $E\{\hat{\underline{x}}(t) - \underline{x}(t)\}$ of (3.66) is identically zero. Hence

$$E\{\hat{\underline{x}}(t)\} = E\{\underline{x}(t)\} \quad \text{for all } t. \tag{3.67}$$

Or, using the previous notations:

$$\underline{\mu}_{\hat{x}}(t) = \underline{\mu}_{x}(t) . \tag{3.68}$$

<u>Property 3.2</u>: The average values of the trajectory sensitivity and the estimate sensitivity of the compensated system are equal when modelling errors are zero.

<u>Proof</u>: Subtracting (3.56) from (3.57) and rearranging, one has

$$(\dot{\hat{\underline{\xi}}} - \dot{\underline{\xi}}) = [F - KH](\hat{\underline{\xi}} - \underline{\xi}) + [F_\alpha - K_\alpha H][\hat{\underline{x}}(\alpha_n,t)-\underline{x}(\alpha_n,t)]. \tag{3.69}$$

Taking the expected value of (3.69), assuming that the order of taking the expectation and the differentiation can be interchanged, one has

$$\frac{d}{dt} E\{\hat{\underline{\xi}}-\underline{\xi}\} = [F-KH]E\{\hat{\underline{\xi}}-\underline{\xi}\} + [F_\alpha-K_\alpha H][E\{\hat{\underline{x}}(\alpha_n,t\} - E\{\underline{x}(\alpha_n,t)\}]. \tag{3.70}$$

Using the unbiased property from (2.31), (3.70) becomes

$$\frac{d}{dt} E\{\hat{\underline{\xi}} - \underline{\xi}\} = [F - KH]E\{\hat{\underline{\xi}} - \underline{\xi}\}. \tag{3.71}$$

Since

$$E\{\hat{\underline{\xi}}(t_o) - \underline{\xi}(t_o)\} = 0$$

the solution of (3.71) is identically zero, i.e.,

$$E\{\hat{\underline{\xi}}(t)\} = E\{\underline{\xi}(t)\} \text{ for all } t. \tag{3.72}$$

Or in terms of the notations previously defined,

$$\underline{\mu}_{\hat{\xi}}(t) = \underline{\mu}_{\xi}(t) \quad \text{for all } t\,. \tag{3.73}$$

The relationships just established in Properties 3.1 and 3.2 for the compensated system are analogous to those obtained earlier for the original (uncompensated) LSR. It is now clear that the dimension of (3.61) is reduced by 2n when the relationships in (3.68) and (3.73) are applied to $\underline{\mu}_\theta$ in (3.61). Therefore, the total number of equations involved in the TPBVP is now 2n(8n+5).

Property 3.3: When there are no modelling errors, the compemsated system reduces on the average to the original nominal LSR.

Proof: Consider (2.24) of the original nominal LSR.

$$\dot{\underline{x}} = F(\alpha_n)\underline{x} - G(\alpha_n)\underline{u}(t) + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o. \tag{2.24}$$

Taking the expected value of (2.24) and using Property 2.2, one has

$$\frac{d}{dt} E\{\underline{x}(t)\} = F(\alpha_n)E\{\underline{x}(t)\} + G(\alpha_n)\underline{u}_d^*(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o, \quad (3.74)$$

assuming that the order of taking the expectation and differentiation can be interchanged. Now, consider (3.53) of the compensated system with $\alpha = \alpha_n$.

$$\dot{\underline{x}} = F(\alpha_n)\underline{x} - G(\alpha_n)S(t)\hat{\underline{x}} + G(\alpha_n)[\underline{u}_d^*(t)+S\underline{x}_d^*(t)] + \underline{w}(t), E\{\underline{x}(t_o)\} = \underline{x}_o \quad (3.53)$$

Taking the expected value of (3.53) with $\alpha = \alpha_n$ and assuming that the order of taking the expectation and the differentiation can be interchanged, one has

$$\frac{d}{dt} E\{\underline{x}(t)\} = F(\alpha_n)E\{\underline{x}(t)\} - G(\alpha_n)S(t)E\{\hat{\underline{x}}(t)\} + G(\alpha_n)[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)], \quad E\{\underline{x}(t_o)\} = \underline{x}_o. \quad (3.75)$$

Using Property 2.1 in (3.75), one is left with

$$\frac{d}{dt} E\{\underline{x}(t)\} = F(\alpha_n)E\{\underline{x}(t)\} + G(\alpha_n)\underline{u}_d^*(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o. \quad (3.76)$$

It is clear that equations (3.74) and (3.76) are the same; and hence the results. Property 3.3 indicates that the average response of the compensated system is the same as the average response of the original LSR, when there are no modelling errors.

Property 3.4: The average values of the optimal nominal control of the original LSR and the optimal nominal control of the compensated system are equal.

Proof: Taking the expected value of (3.52) with $\alpha = \alpha_n$, which defines the optimal nominal control of the compensated system, one has

$$E\{\underline{u}[\hat{\underline{x}}(\alpha_n,t),t]\} = \underline{u}_d^*(t) + S(t)[\underline{x}_d^*(t) - E\{\hat{\underline{x}}(\alpha_n,t)\}],$$

using Property 2.1, $\quad = \underline{u}_d^*(t)$. (3.77)

From Property 2.2, one has

$$E\{\underline{u}(\alpha_n,t)\} = \underline{u}_d^*(t) , \tag{3.78}$$

where $\underline{u}(\alpha_n,t)$ is the nominal optimal control of the original LSR. Comparing (3.77) and (3.78), one has

$$E\{\underline{u}[\hat{\underline{x}}(\alpha_n,t),t]\} = E\{\underline{u}(\alpha_n,t)\} \tag{3.79}$$

Hence, the desired results.

Property 3.5: The mean square deviation of the nominal control of the compensated system, $\underline{u}[\hat{\underline{x}}(\alpha_n,t),t]$, from the optimal nominal control of the original LSR, $\underline{u}(\alpha_n,t)$, is given by

$$[S(t) - C(\alpha_n,t)]\cdot \mathrm{Var}[\hat{\underline{x}}(\alpha_n,t)]\cdot [S(t) - C(\alpha_n,t)]^T .$$

From (2.26), the optimal nominal control of the original LSR is

$$\underline{u}(\alpha_n,t) = -C(t)\hat{\underline{x}}(\alpha_n,t) \tag{3.80}$$

From (3.52), the nominal control of the compensated system is

$$\underline{u}[\hat{\underline{x}}(\alpha_n,t),t] = \underline{u}_d^*(t) + S(t)[\underline{x}_d^*(t) - \hat{\underline{x}}(\alpha_n,t)] \ . \qquad (3.81)$$

Define $\qquad \underline{e}_u(t) = \underline{u}[\hat{\underline{x}}(\alpha_n,t)] - \underline{u}(\alpha_n,t).$

Using (3.80) and (3.81),

$$\underline{e}_u(t) = \underline{u}_d^*(t) + S(t)\underline{x}_d^*(t) - S(t)\hat{\underline{x}}(\alpha_n,t) + C(t)\hat{\underline{x}}(\alpha_n,t),$$

Using (2.3), $= -C(t)\underline{x}_d^*(t) + S(t)\underline{x}_d^*(t) - S(t)\hat{\underline{x}}(\alpha_n,t)+C(t)\hat{\underline{x}}(\alpha_n,t),$

$$= -[S(t) - C(t)][\hat{\underline{x}}(\alpha_n,t) - \underline{x}_d^*(t)][S(t)-C(t)]^T.$$

Hence

$$E\{\underline{e}_u(t)\underline{e}_u^T(t)\} = [S(t)-C(t)]E\{[\hat{\underline{x}}(\alpha_n,t)-\underline{x}_d^*(t)][\hat{\underline{x}}(\alpha_n,t) - \underline{x}_d^*(t)]^T\}[S(t)-C(t)]^T, \qquad (3.82)$$

Using Property 2.1, $= [S(t)-C(t)]E\{[\hat{\underline{x}}(\alpha_n t)-E\{\hat{\underline{x}}(\alpha_n,t)\}] \cdot [\hat{\underline{x}}(\alpha_n,t)-E\{\hat{\underline{x}}(\alpha_n,t)\}]^T\}[S(t)-C(t)]^T,$

$$= [S(t)-C(t)]\mathrm{Var}[\hat{\underline{x}}(\alpha_n,t)][S(t)-C(t)]^T \qquad (3.83)$$

Hence the desired results. It can be seen from (3.83) that the mean square deviation of the nominal control of the compensated system from the optimal nominal control of the original LSR is a function of S(t), C(t) and Var[$\hat{\underline{x}}(\alpha_n,t)$]. For a given LSR, C(t) is fixed, S(t) varies depending on the weighting matrices Q_1, Q_2 and Q_f in (3.55), and in general Var[$\hat{\underline{x}}(\alpha_n,t)$] is small, bounded (because the Kalman filter is assumed to be stable under the nominal conditions) and dependent on the process and the measurement noises. Therefore, the extent to which $\underline{u}[\hat{\underline{x}}(\alpha_n,t)]$ deviates from $\underline{u}(\alpha_n,t)$ depends on the noises and on how much the trajectory sensitivity is penalized in (3.55). This knowledge helps to explain how the cost J, see (3.36), of the compensated system differs from the cost J of the original LSR when there are no modelling errors. The term $E\{\|\underline{x}(t)\|_A^2\}$ in the integrand of the cost J should remain nearly unchanged for the original LSR and for the compensated system, because the compensated system is formed as a result of minimizing $E\{\|\Delta x(t)\|^2\}$. Therefore, the difference between the cost J of the compensated system and that of the original LSR is mainly contributed by the difference in the second term of the integrand, i.e., the mean square control. Since Property 3.5 gives an expression for the mean square deviation of the nominal control of the compensated system from the optimal nominal control of the original LSR, it can be used to an advantage as to how the cost J of the compensated system differs from that of the original LSR.

The procedure involved in applying the compensation method to reduce the trajectory sensitivity of the LSR can be summarized as follows. A two-degree-of-freedom control structure is formed according to (3.52). Then, $S(t)$ is chosen such that J_s in (B.35) is minimized subject to (B.24) and (B.32). An application of the Matrix Minimum Principle to this auxiliary minimization problem yields the necessary conditions given by (3.60) - (3.64). These necessary conditions represent a TPBVP which will be numerically solved off-line by a steepest-descent algorithm. The number of equations involved in the resulting TPBVP is $2n(8n+5)$ for a scalar parameter, where n is the plant order.

<u>Extension of the Compensation Method to a Multi-Parameter and Multi-Variable Problem</u>

The compensation method can be extended in a straightforward manner to the case where $\underline{\alpha}$ is a parameter m-vector, i.e.,

$$\underline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T. \qquad (3.84)$$

The auxiliary minimization problem defined by (B.35), (B.24) and (B.32) now becomes

$$\min_{S(t)} J_s = \min_{S(t)} \operatorname{tr}\left\{ \sum_{i=1}^{m} M^i P^i_{\theta\theta}(t_f) + \int_{t_0}^{t_f} \sum_{i=1}^{m} [Q^i(t)+R^i(t)] P^i_{\theta\theta}(t)dt \right\} \qquad (3.85)$$

subject to

$$\dot{P}^i_{\theta\theta} = D^i P^i_{\theta\theta} + P^i_{\theta\theta} D^{i^T} + \Gamma^i(\underline{\mu}^i_\theta),\ P^i_{\theta\theta}(t_0) = Q^i_0,\ i=1,2,\ldots,m; \tag{3.86}$$

$$\dot{\underline{\mu}}^i_\theta = D^i \underline{\mu}_\theta + \underline{\bar{\beta}}^i(t),\ \underline{\mu}^i_\theta(t_0) = \underline{\mu}^i,\ i=1,2,\ldots,m; \tag{3.87}$$

where

$$P^i_{\theta\theta}(t) \triangleq E\{\underline{\theta}^i(t)[\underline{\theta}^i(t)]^T\}, \tag{3.88}$$

$$\underline{\mu}^i_\theta(t) \triangleq E\{\underline{\theta}^i(t)\}, \tag{3.89}$$

$$\underline{\theta}^i(t) \triangleq \begin{bmatrix} \underline{\xi}^i(t) \\ \hat{\underline{\xi}}^i(t) \\ \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix}, \tag{3.90}$$

$$\underline{\xi}^i(t) \triangleq \left.\frac{\partial \underline{x}(\underline{\alpha},t)}{\partial \alpha_i}\right|_{\underline{\alpha}=\underline{\alpha}_n}, \tag{3.91}$$

$$\hat{\underline{\xi}}^i(t) \triangleq \left.\frac{\partial \hat{x}(\underline{\alpha},t)}{\partial \alpha_i}\right|_{\underline{\alpha}=\underline{\alpha}_n}, \tag{3.92}$$

$$M^i \triangleq \begin{bmatrix} Q^i_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.93}$$

$$Q^i(t) \triangleq \begin{bmatrix} Q_1^i(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.94}$$

$$R^i(t) \triangleq \begin{bmatrix} S^T(t)Q_2^i(t)S(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.95}$$

$$D^i \triangleq \begin{bmatrix} F & -GS & F_{\alpha_i} & -G_{\alpha_i}S \\ KH & F-GS-KH & K_{\alpha_i} & F_{\alpha_i}-K_{\alpha_i}H-G_{\alpha_i}S \\ 0 & 0 & F & -GS \\ 0 & 0 & KH & F-GS-KH \end{bmatrix}, \tag{3.96}$$

$$\Gamma^i(\underline{\mu}_\theta^i) \triangleq \begin{bmatrix} \Gamma_{11}^i & \Gamma_{12}^i & \Gamma_{13}^i & \Gamma_{14}^i \\ \Gamma_{12}^{i^T} & \Gamma_{22}^i & \Gamma_{23}^i & \Gamma_{24}^i \\ \Gamma_{13}^{i^T} & \Gamma_{23}^{i^T} & \Gamma_{33}^i & \Gamma_{34}^i \\ \Gamma_{14}^{i^T} & \Gamma_{24}^{i^T} & \Gamma_{34}^{i^T} & \Gamma_{44}^i \end{bmatrix}, \tag{3.97}$$

$$\Gamma_{11}^{i} \triangleq M_2^i \underline{\mu}_\xi^{i^T} + \underline{\mu}_\xi^i M_2^{i^T}, \quad \Gamma_{12}^{i} \triangleq M_2^i \underline{\mu}_\xi^{i^T} + \underline{\mu}_\xi^i M_2^{i^T},$$

$$\Gamma_{13}^{i} \triangleq M_2^i \underline{\mu}_x^{i^T} + \underline{\mu}_\xi^i M_1^{i^T}, \quad \Gamma_{14}^{i} \triangleq M_2^i \underline{\mu}_x^{i^T} + \underline{\mu}_\xi^i M_1^{i^T},$$

$$\Gamma_{22}^{i} \triangleq M_2^i \underline{\mu}_\xi^{i^T} + \underline{\mu}_\xi^i M_2^{i^T} + K_{\alpha_i} Q_v K_{\alpha_i}^T, \quad \Gamma_{23}^{i} \triangleq M_2^i \underline{\mu}_x^{i^T} + \underline{\mu}_\xi^i M_1^{i^T},$$

$$\Gamma_{24}^{i} \triangleq M_2^i \underline{\mu}_x^{i^T} + \underline{\mu}_\xi^i M_1^{i^T} + K_{\alpha_i} Q_v K^T, \quad \Gamma_{33}^{i} \triangleq M_1^i \underline{\mu}_x^{i^T} + \underline{\mu}_x^i M_1^{i^T} + Q_w,$$

$$\Gamma_{34}^{i} = M_1^i \underline{\mu}_x^{i^T} + \underline{\mu}_x^i M_1^{i^T}, \quad \Gamma_{44}^{i} \triangleq M_1^i \underline{\mu}_x^{i^T} + \underline{\mu}_x^i M_1^{i^T} + KQ_v K^T, \quad (3.98)$$

$$M_2^i \triangleq G_{\alpha_i}[\underline{u}_d^*(t) + S(t)\underline{x}_d^*(t)], \quad (3.99)$$

$$\underline{\overline{\beta}}^i(t) \triangleq \begin{bmatrix} M_2^i \\ M_2^i \\ M_1^i \\ M_1^i \end{bmatrix}. \quad (3.100)$$

The superscript i corresponds to the parameter component α_i while the subscript α_i denotes a partial derivative with respect to α_i. The necessary conditions previously obtained in (3.60) - (3.64) now become

$$\dot{\Lambda}^i(t) = -D^{i^T}\Lambda^i(t) - \Lambda^i(t)D^i - Q^i - R^i(S), \quad \Lambda^i(t_f) = M^i,$$

$$i = 1,2,\ldots,m \; ; \quad (3.101)$$

$$\dot{\underline{\lambda}}^i(t) = -D^{i^T}\underline{\lambda}^i - 2\Lambda^i\underline{\overline{\beta}}^i, \quad \underline{\lambda}^i(t_f) = \underline{0}, \; i=1,2,\ldots,m; \quad (3.102)$$

$$\dot{P}^i_{\theta\theta}(t) = D^i P^i_{\theta\theta}(t) + P^i_{\theta\theta}(t) D^{i^T} + \Gamma^i, \; P^i_{\theta\theta}(t_o) = Q^i_o,$$
$$i=1,2,\ldots,m; \quad (3.103)$$

$$\dot{\underline{\mu}}^i_\theta(t) = D^i \underline{\mu}^i_\theta + \overline{\underline{\beta}}^i, \quad \underline{\mu}^i_\theta(t_o) = \underline{\mu}^i_o, \quad i=1,2,\ldots,m; \quad (3.104)$$

$$\frac{\partial\hat{H}}{\partial S} = \sum_{i=1}^{m} \hat{H}^i_s = 0 \;, \quad (3.105)$$

where

$$\begin{aligned}
\hat{H}^i_s = {} & 2Q^i_2 SP^i_{\xi\xi} - 2G^T[(\Lambda^i_{\xi\xi} + \Lambda^{i^T}_{\xi\hat{\xi}})P^i_{\xi\hat{\xi}} + (\Lambda^{i^T}_{\xi x} + \Lambda^{i^T}_{\xi\hat{x}})P^i_{\xi\hat{x}} + (\Lambda^i_{\xi\hat{\xi}} + \Lambda^i_{\hat{\xi}\hat{\xi}})P^i_{\hat{\xi}\hat{\xi}} \\
& + (\Lambda^{i^T}_{\hat{\xi}x} + \Lambda^{i^T}_{\hat{\xi}\hat{x}})P^i_{\hat{\xi}\hat{x}} + (\Lambda^i_{\xi x} + \Lambda^i_{\hat{\xi}x})P^{i^T}_{\xi\hat{x}} + (\Lambda^i_{xx} + \Lambda^{i^T}_{x\hat{x}})P^i_{x\hat{x}} \\
& + (\Lambda^i_{\xi\hat{x}} + \Lambda^i_{\hat{\xi}\hat{x}})P^{i^T}_{\hat{\xi}\hat{x}} + (\Lambda^i_{x\hat{x}} + \Lambda^i_{\hat{x}\hat{x}})P^i_{\hat{x}\hat{x}}] - 2G^T_{\alpha_i}[(\Lambda^i_{\xi\xi} + \Lambda^{i^T}_{\xi\hat{\xi}})P^i_{\xi\hat{x}} \\
& + (\Lambda^i_{\xi\hat{\xi}} + \Lambda^i_{\hat{\xi}\hat{\xi}})\, P^i_{\hat{\xi}\hat{x}} + (\Lambda^i_{\xi x} + \Lambda^i_{\hat{\xi}x})P^i_{x\hat{x}} + (\Lambda^i_{\xi\hat{x}} + \Lambda^i_{\hat{\xi}\hat{x}})P^i_{\hat{x}\hat{x}}] \\
& + 2G^T[(\Lambda^{i^T}_{\xi x} + \Lambda^{i^T}_{\xi\hat{x}})\underline{\mu}^i_\xi + (\Lambda^{i^T}_{\hat{\xi}x} + \Lambda^{i^T}_{\hat{\xi}\hat{x}})\underline{\mu}^i_{\hat{\xi}} + (\Lambda^i_{xx} + \Lambda^{i^T}_{x\hat{x}})\underline{\mu}^i_x \\
& + (\Lambda^i_{x\hat{x}} + \Lambda^i_{\hat{x}\hat{x}})\underline{\mu}^i_{\hat{x}}]\underline{x}^{*T}_d] + 2G^T_{\alpha_i}[(\Lambda^i_{\xi\xi} + \Lambda^{i^T}_{\xi\hat{\xi}})\underline{\mu}^i_{\hat{\xi}} + (\Lambda^i_{\xi\hat{\xi}} + \Lambda^i_{\hat{\xi}\hat{\xi}})\underline{\mu}^i_{\hat{\xi}} \\
& + (\Lambda^i_{\xi x} + \Lambda^i_{\hat{\xi}x})\underline{\mu}^i_x + (\Lambda^i_{\xi\hat{x}} + \Lambda^i_{\hat{\xi}\hat{x}})\underline{\mu}^i_{\hat{x}}]\underline{x}^{*T}_d] - G^T[\underline{\mu}^i_\xi(\underline{\lambda}^{i^T}_\xi + \underline{\lambda}^{i^T}_{\hat{\xi}}) \\
& + \underline{x}^*_d(\underline{\lambda}^{i^T}_x + \underline{\lambda}^{i^T}_{\hat{x}}) - (\underline{\lambda}^i_x + \underline{\lambda}^i_{\hat{x}})\underline{x}^{*T}_d] - G^T_{\alpha_i}[\underline{x}^*_d(\underline{\lambda}^{i^T}_\xi + \underline{\lambda}^{i^T}_{\hat{\xi}}) \\
& - (\underline{\lambda}^i_\xi + \underline{\lambda}^i_{\hat{\xi}})\underline{x}^{*T}_d]. \qquad (3.106)
\end{aligned}$$

At a first glance, the number of equations involved in the resulting TPBVP given by (3.101) - (3.104) appears to be $4nm(4n+3)$ equations. However, the number of equations can be reduced quite considerably if the relationships established by Properties 3.1 and 3.2 and certain facts are noted as follows. (It is still assumed that the steepest-descent gradient algorithm is being used for a numerical solution of the TPBVP.) First, consider equation (3.104). It is obvious from Properties 2.1, 3.1 and 3.4 that the sub-vectors, $\underline{\mu}_x$ and $\underline{\mu}_{\hat{x}}$, of $\underline{\mu}_\theta$ can be expressed as

$$\underline{\mu}_x\ t) = \underline{\mu}_{\hat{x}}(t) = \underline{x}_d^*(t), \text{ for all } t, \tag{3.107}$$

where $\underline{x}_d^*(t)$ is the optimal nominal trajectory of the LDR. Since $\underline{x}_d^*(t)$ is independent of $S(t)$ and dependent only on the nominal parameter values, $\underline{\alpha}_n$, there is no need to solve (3.104) for $\underline{\mu}_x$ and $\underline{\mu}_{\hat{x}}$ for different parameter components: $\alpha_1, \alpha_2, \ldots, \alpha_m$. In other words, the pre-calculated $\underline{x}_d^*(t)$ can be used as a part of a solution of (3.104) as well as in the two-degree of-freedom control in (3.52). Therefore the number of equations involved in solving (3.104) for m parameter components is now 2nm. However, a further analysis based on Property 3.2 reveals that in any i^{th} iteration of the gradient algorithm,

$$\underline{\mu}_\xi^j(t) = \underline{\mu}_{\hat{\xi}}^j(t), \quad \text{for all } t, \tag{3.108}$$

where the superscript j corresponds to the parameter component α_j. Eventually, the final number of equations involved in solving (3.104) is nm instead of 4nm per iteration, assuming that $\underline{x}_d^*(t)$ is available. Second, consider equation (3.103). A careful inspection of (3.103) reveals that in any i^{th} iteration of the gradient algorithm the following is true:

$$P^1_{xx}(t) = P^2_{xx}(t) = \dots = P^m_{xx}(t) \quad \text{for all } t, \tag{3.109}$$

$$P^1_{x\hat{x}}(t) = P^2_{x\hat{x}}(t) = \dots = P^m_{x\hat{x}}(t) \quad \text{for all } t, \tag{3.110}$$

$$P^1_{\hat{x}\hat{x}}(t) = P^2_{\hat{x}\hat{x}}(t) = \dots = P^m_{\hat{x}\hat{x}}(t) \quad \text{for all } t. \tag{3.111}$$

Therefore, the number of equations involved in solving (3.103) is $n(2(3m+1)n + m + 1)$ instead of $2nm(4n + 1)$ per iteration. For example, for $n = 2$, $m = 2$, the number of equations is 62 instead of 72. A considerable saving is envisaged when n and m are large. Finally, since the number of equations involved in solving the costate equations (3.101) and (3.102) is $2nm(4n + 3)$, the total number of equations that have to be solved is $nm + n(2(3m+1)n + m + 1) + 2nm(4n+3) = (14m+2)n^2+(8m+1)n$ per iteration.

A further analysis will reveal, in addition, the fact that all the equations involved in the TPBVP need not be solved simultaneously. Equation (3.85) can be written as

$$\min_{S(t)} J_s = \min_{S(t)} \sum_{i=1}^{m} J_s^i , \tag{3.112}$$

where

$$J_s^i \triangleq \mathrm{tr}\left\{M^i P_{\theta\theta}^i(t_f) + \int_{t_0}^{t_f} [Q^i(t) + R^i(t)] P_{\theta\theta}^i(t)dt\right\} . \tag{3.113}$$

Since for a given S(t) in a given iteration $\underline{\xi}^i(t)$ and $\hat{\underline{\xi}}^i(t)$ are independent of $\underline{\xi}^j(t)$ and $\hat{\underline{\xi}}^j(t)$, respectively, for $i \neq j$ and $i,j = 1,2,\ldots,m$; the sub-matrix elements of $P_{\theta\theta}^i$ that are related to $\underline{\xi}^i$ or $\hat{\underline{\xi}}^i$ are independent of the corresponding sub-matrix elements of $P_{\theta\theta}^j$ that are related to $\underline{\xi}^j$ or $\hat{\underline{\xi}}^j$. Or, symbolically, $P_{\xi\xi}^i$, $P_{\xi\hat{\xi}}^i$, $P_{\xi x}^i$ $P_{\xi\hat{x}}^i$, $P_{\hat{\xi}\hat{\xi}}^i$, $P_{\hat{\xi}x}^i$, $P_{\hat{\xi}\hat{x}}^i$, are independent of $P_{\xi\xi}^j$, $P_{\xi\hat{\xi}}^j$, $P_{\xi x}^j$, $P_{\xi\hat{x}}^j$, $P_{\hat{\xi}\hat{\xi}}^j$, $P_{\hat{\xi}x}^j$, $P_{\hat{\xi}\hat{x}}^j$, respectively, for $i \neq j$, i, $j = 1,2,\ldots,m$. Similarly, for a given S(t) in a given iteration, Λ^i and $\underline{\lambda}^i$ are independent of Λ^j and $\underline{\lambda}^j$, for $i \neq j$, $i,j = 1,2,\ldots,m$, respectively. With these facts in mind, in any $m^{\underline{th}}$ iteration, the following steps can be taken to numerically solve the TPBVP by means of the gradient algorithm. Step (a): assuming that $S^m(t)$ is known either guessed or calculated from the previous $(m-1)^{\underline{th}}$ iteration. Step (b): equation (3.104) is solved for $\underline{\mu}_\theta^1$ which corresponds to parameter component α_1; this involves 2n equations. Then, equation (3.103) is solved for $P_{\theta\theta}^1(t)$ which corresponds to the parameter component α_1; this involves 2n(4n+1) equations. Then, equation (3.101) is solved for

$\Lambda^1(t)$ which corresponds to the parameter component α_1; this involves $2n(4n+1)$ equations. Then, equation (3.102) is solved for $\underline{\lambda}^1(t)$ which corresponds to α_1; this involves $4n$ equations. Then, $\hat{H}_s^i(t)$ is calculated according to (3.106) but with $i = 1$. And finally for this parameter component, α_1, J_s^i is calculated according to (3.113) but with $i = 1$. Step (c): the calculations similar to those in Step (b) are repeated for the parameter components: $\alpha_2, \alpha_3, \ldots, \alpha_m$, respectively, bearing in mind the useful relationships given by (3.107) - (3.111). In other words, $\underline{\mu}_\theta^j$, $P_{\theta\theta}^j$, Λ^j, $\underline{\lambda}^j$, $\hat{H}_s^j$ and J_s are calculated for $j=2,3,\ldots,m$, respectively. Step (d): calculate $\frac{\partial \hat{H}}{\partial S}$ and J_s according to (3.105) and (3.112), respectively. Then, update $S^m(t)$, if needed for use in the next iteration according to

$$S^{m+1}(t) = S^m(t) - k\,\frac{\partial \hat{H}}{\partial S}$$

Step (e): if J_s has not reached its minimum value, repeat steps (b), (c) and (d) with the new $S^{m+1}(t)$.

It is clear that if the procedure just described for the TPBVP is followed, in no time one will have to solve more than $2n(4n+1)$ equations at one time. A reduced number of equations helps to reduce the adverse effects of rounding-off errors and also the computer memory core requirement.

Summary

The compensation method presented in this section is an off-line scheme which is based on the sensitivity concept. It employs a two-degree-of-freedom control structure involving a compensating feedback gain matrix, S(t), which is chosen such that the sensitivity performance index, J_s, containing the expected value of quadratic terms in the trajectory sensitivity is minimized. This problem is stochastic in nature. However, for the problem on hand, it can be converted to its equivalent deterministic optimization problem for which the necessary conditions can be obtained via the Matrix Minimum Principle. These necessary conditions represent a TPBVP involving $(14m + 2)n^2 + (8m + 1)n$ equations, where n and m are the plant order and the number of parameters, respectively. The TPBVP is numerically solved by the steepest-descent gradient algorithm in order to obtain the optimal S(t). Then, with S(t) thus obtained, the compensated system is connected up to form a simple configuration as shown in the block diagram in Figure 5. The compensated system has several desirable features. First, it is an off-line scheme with a fairly simple configuration. Second, the two-degree-of-freedom control structure enables a reduction in the trajectory sensitivity to be effected independently of the optimal trajectory. This means that both the desired trajectory sensitivity and the desired optimal trajectory can

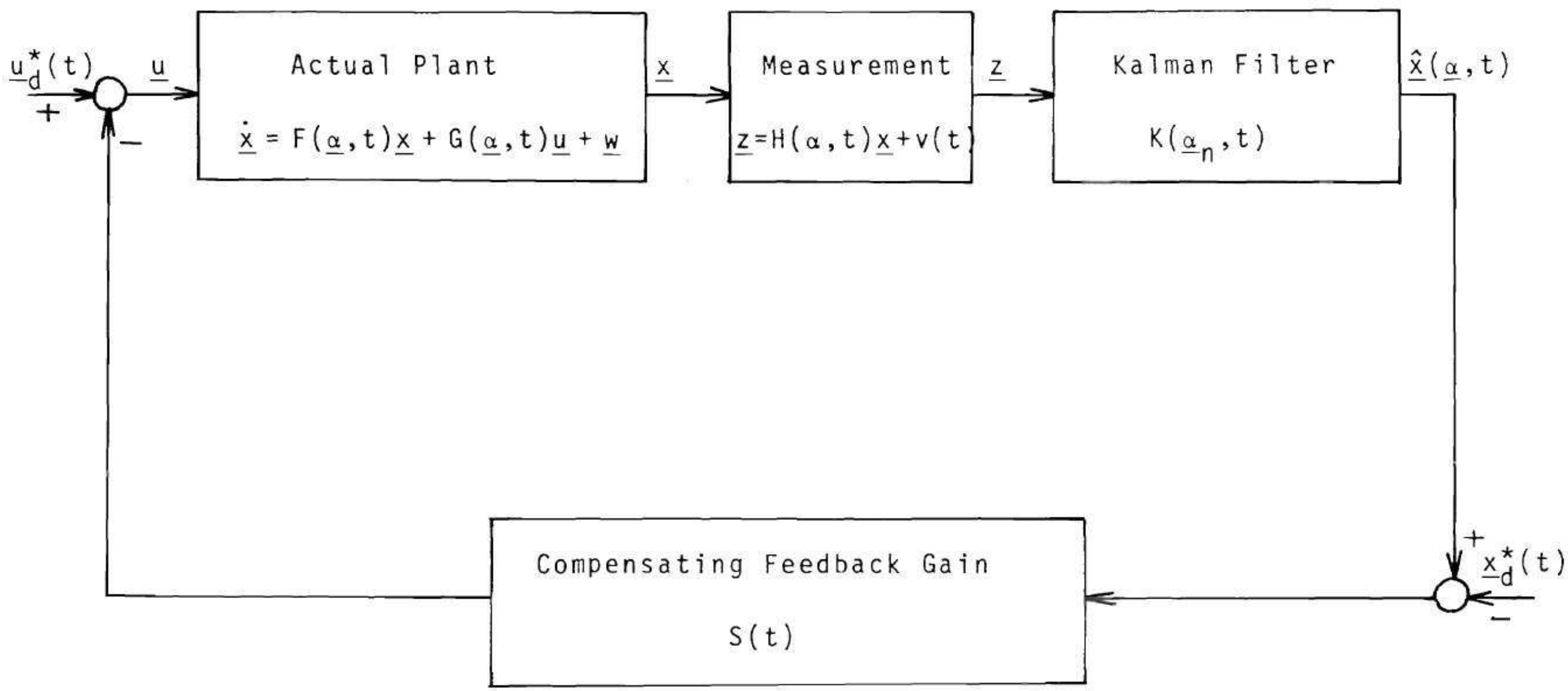

Figure 5. The Compensated System.

be independently met. Third, on the average the compensated system reduces to the original LSR when there are no parameter mismatches. Fourth, it shares an advantage with the other compensation methods which are based on the sensitivity concept in that the knowledge of the parameter errors, $\delta\underline{\alpha} = \underline{\alpha} - \underline{\alpha}_n$, is not required; this eliminates the need for parameter estimation. Fifth, the trajectory sensitivity components can be selectively reduced by a proper choice of the weight matrices, $Q^i(t)$ and $R^i(t)$, $i = 1,2,\ldots,m$; see (3.85). In general, no analytical methods of obtaining suitable values of $Q^i(t)$ and $R^i(t)$ are available; they have to be obtained by trial and error. Finally, in the compensated system, the mean values of the state $\underline{x}(t)$ and the estimate $\hat{\underline{x}}(t)$ are equal; and so are the mean values of the trajectory sensitivity $\underline{\xi}(t)$ and the estimate sensitivity $\hat{\underline{\xi}}(t)$. These relationships are analogous to those in the original LSR. They help to reduce the number of equations to be solved. An undesirable feature of the compensation method is that it results in a fairly large number of equations. This may limit its use to low-order systems only. The large number of equations is a direct consequence of the complexity due to the stochastic nature of the problem as well as the increased order of the combined dynamics of the plant and the Kalman filter. The number of equations of the TPBVP of the compensation method by the author is comparable to that of the compensation method by Stavroulakis and Sarachik in [50].

In the foregoing, the compensation method has been formulated on the basis of the first order sensitivity only. It should be noted, however, that it can be generalized to include higher-order sensitivity terms, of course, at a greater amount of computation. If this is done, the assumption of small parameter variations can be relaxed somewhat.

CHAPTER IV

SIMULATION EXAMPLES

In this chapter, the workability, usefulness and effectiveness of the compensation method will be demonstrated via four simulation examples. The first two examples are of first order whereas the last two are of second order. In the course of presenting these examples, whenever possible, the analytic solutions to the problems are given in order that some insight into the problems might be gained. A fourth-order Runge-Kutta method is used for all numerical integrations, and numerical interpolations are used where necessary.

Example 1

First order LSR's are used in many practical applications such as the control of the angular velocity of a D.C. motor [58, p. 189], the control of the speed of a wire being wound on a reel-winding mechanism [58, p. 234], and so on. In this example, the compensation method will be used to reduce the trajectory sensitivity of a first-order LSR due to the modelling error in the plant time constant. Consider the LSR problem defined by

$$\min_{u(t)} J = \min_{u(t)} \frac{1}{2} E\{\int_0^4 [x^2(t) + u^2(t)]dt\} , \tag{4.1}$$

subject to

$$\dot{x} = -\alpha x + u + w, \ E\{x(t_0)\} = 5, \ var\{x\ (t_0)\} = 0.02; \tag{4.2}$$

$$z = x + v , \tag{4.3}$$

where α is the parameter of which the nominal value is $\alpha_n = 1.0$, w and v are independent white Gaussian noises with the following properties:

$$E\{w(t)\} = E\{v(t)\} = 0 , \tag{4.4}$$

$$E\{w(t)w^T(\tau)\} = 0.01 , \tag{4.5}$$

$$E\{v(t)V^T(\tau)\} = 0.015 . \tag{4.6}$$

<u>Step I</u>: Identifications of Variables.

With reference to (2.12) - (2.19) and (2.23), the following identifications are made.

$\alpha_n = 1.0, \ s_f = 0, \ a = 1.0, \ b = 1.0,$

$f = -\alpha, \quad g = 1.0, \quad h = 1.0,$

$x_o = 5.0, \quad p_o = 0.02,$

$q_w = 0.01, \quad q_v = 0.015,$

$t_o = 0, \quad t_f = 4$, step size of numerical integration = .02 second.

Step II: Sensitivity Analysis of the Original LSR

(a) The Deterministic Controller. Using (2.5) and (2.6), the deterministic controller gain, c(t), is given by

$$\dot{c}(t) = 2\alpha c(t) + c^2(t) - 1, \quad c(4) = 0 \tag{4.7}$$

The numerical solution of (4.7) with $\alpha = \alpha_n$ is shown in Figure 6. c(t) is stored for later use. c(t) remains virtually constant for $0 \leq t \leq 2.0$. In general, the analytical solution for c(t) is not available; however, for this particular case it is available and given below.

$$c(t) = \frac{(\sqrt{\alpha^2 + 1} - \alpha) + (\sqrt{\alpha^2 + 1} + \alpha)\dfrac{\alpha - \sqrt{\alpha^2 + 1}}{\alpha + \sqrt{\alpha^2 + 1}}\, e^{-2\sqrt{\alpha^2+1}(4-t)}}{1 - \dfrac{\alpha - \sqrt{\alpha^2 + 1}}{\alpha + \sqrt{\alpha^2 + 1}}\, e^{-2\sqrt{\alpha^2 + 1}(4-t)}},$$

$$0 \leq t \leq 4 . \tag{4.8}$$

Using (4.8) with $\alpha = \alpha_n$, c(0) and c(4) are calculated and found to be the same as the numerical solution in Figure 6. The parameter α appears in the coefficients and in the exponent of both the numerator and the denominator of the expression for c(t) in (4.8).

(b) Sensitivity of the Controller Gain. Using (2.41) and (2.42), the controller gain sensitivity, $c_\alpha(t)$, is given below by

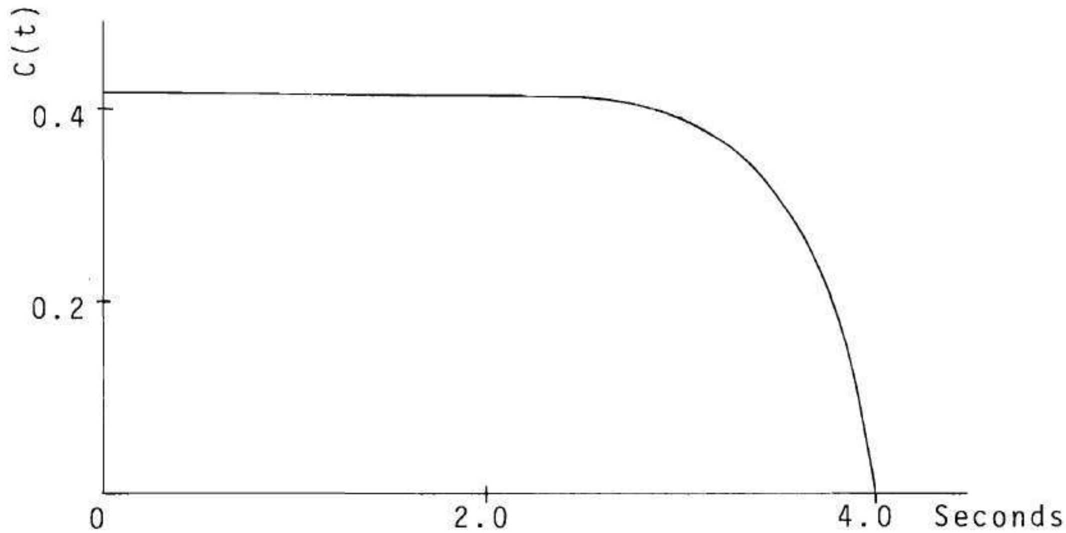

Figure 6. Deterministic Controller Gain, Example 1.

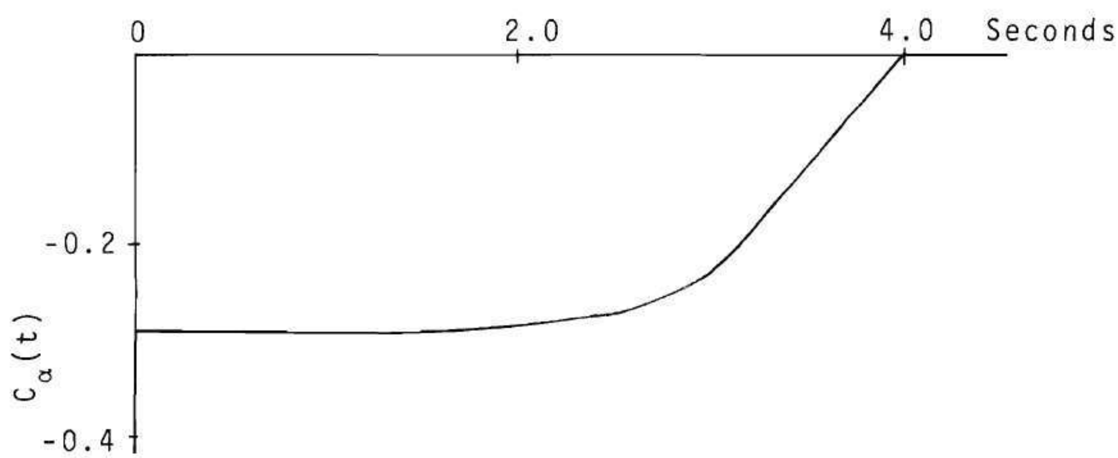

Figure 7. Sensitivity of Deterministic Controller Gain to α, Example 1.

$$\dot{c}_\alpha = 2(\alpha_n\ c(t))c_\alpha + 2c(t),\ c_\alpha(r) = 0. \tag{4.9}$$

The numerical solution of (4.9) with $\alpha = \alpha_n$ is shown in Figure 7. *The analytical expression for $c_\alpha(t)$ can be obtained* by partially differentiating c(t) in (4.8) with respect to α. The result is given below.

$$c_\alpha(t) =$$

$$\frac{\dfrac{\alpha-\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}} + 2(2\alpha(t-4)-1)\dfrac{\alpha-\sqrt{1+\alpha^2}}{\alpha+\sqrt{1+\alpha^2}}\, e^{-2\sqrt{1+\alpha^2}\,(4-t)} - \dfrac{\alpha+\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}}\left[\dfrac{\alpha-\sqrt{1+\alpha^2}}{\alpha+\sqrt{1+\alpha^2}}\right]^2 e^{-4\sqrt{1+\alpha^2}(4-t)}}{1 - \left[\dfrac{\alpha-\sqrt{1+\alpha^2}}{\alpha+\sqrt{1+\alpha^2}}\, e^{-2\sqrt{1+\alpha^2}(4-t)}\right]^2} \tag{4.10}$$

$c_\alpha(0)$ and $c_\alpha(4)$ are calculated using (4.10) with $\alpha = \alpha_n$ and they check against the numerical results of Figure 7. The dependence of $c_\alpha(t)$ on α as indicated by (4.10) is very complicated even for a very simple LSR problem such as this one. Therefore, not much can be said about the behavior of $c_\alpha(t)$ for different values of α and t, except when α becomes very large or very small. Using (4.10), it can be shown that

$$\lim_{\alpha\to\infty} c_\alpha(t) = \lim_{\alpha\to\infty} \frac{1}{\sqrt{1 + \frac{1}{\alpha^2}}} - 1 = 0, \tag{4.11}$$

and similarly that

$$\lim_{\alpha\to-\infty} c_\alpha(t) = 0\,. \tag{4.12}$$

Also, using (4.10) with $\alpha = 0$, one has

$$c_\alpha(t) = \begin{cases} -(1-e^{-2(4-t)})^2/(1+e^{-2(4-t)})^2, & 0 < t < 4 \\ -1 \text{ (approximately)}, & t = 0 \\ 0, & t = 4 \end{cases} \quad (4.13)$$

Hence, the partial behavior of $c_\alpha(t)$ with respect to α and t can be summarized as follows. First, from (4.10), as $t \to 0$ and provided that $2\sqrt{1+\alpha^2}(4-t)$ is sufficiently large, $c_\alpha(t)$ will approach $(\alpha-\sqrt{1+\alpha^2})/\sqrt{1+\alpha^2}$. Second, (4.11) and (4.12) imply that when α becomes either positively or negatively very large, $c_\alpha(t)$ will become smaller and approach zero. Finally, when $\alpha = 0$, $c_\alpha(t)$ is equal to zero at $t = 4$ and gradually approaches -1 as $t \to 0$.

(c) The Kalman Filter. Using (2.22) and (2.23), the Kalman gain, k(t), is given by

$$\dot{k}(t) = -2\alpha k(t) - k^2(t) + \frac{q_w}{q_v}, \quad k(0) = k_o \triangleq \frac{p_o}{q_v} \quad (4.14)$$

k(t) is obtained by a numerical integration of (4.14) with $\alpha=\alpha_n$, and is shown in Figure 8. Store k(t) for later use. An analytical expression for k(t) is also available and given below.

$$k(t) = \frac{(\beta_2 - \alpha) + (\beta_2 + \alpha)\,\dfrac{k_o + \alpha - \beta_2}{k_o + \alpha + \beta_2}\,e^{-2\beta_2 t}}{1 - \dfrac{k_o + \alpha - \beta_2}{k_o + \alpha + \beta_2}\,e^{-2\beta_2 t}}, \quad (4.15)$$

where $\beta_2 \triangleq \sqrt{\alpha^2 + q_w/q_v}$

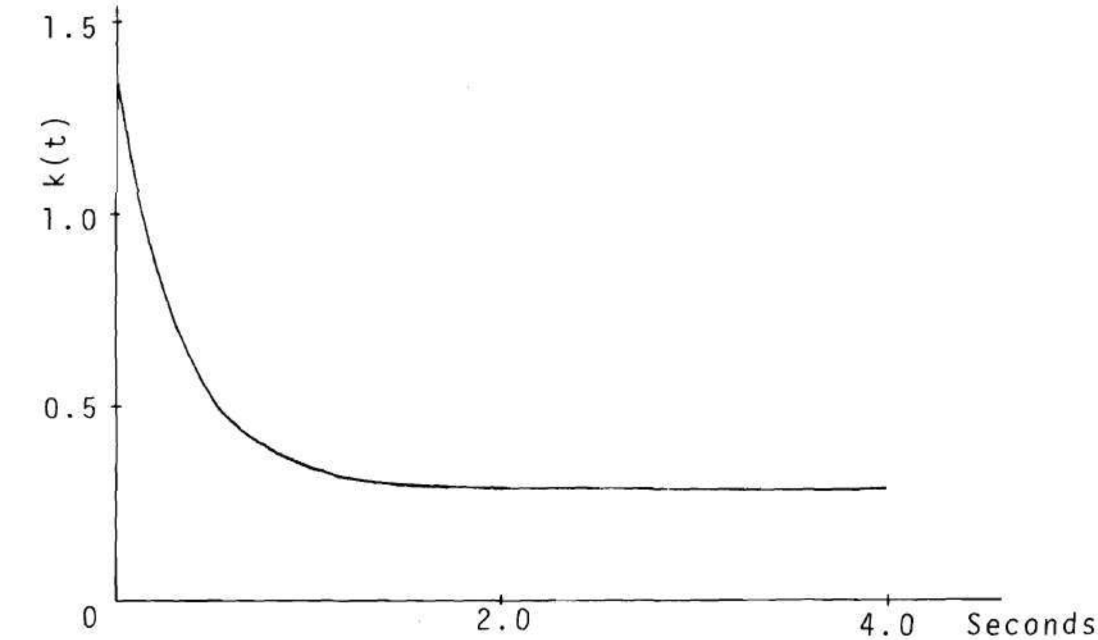

Figure 8. Kalman Gain, Example 1.

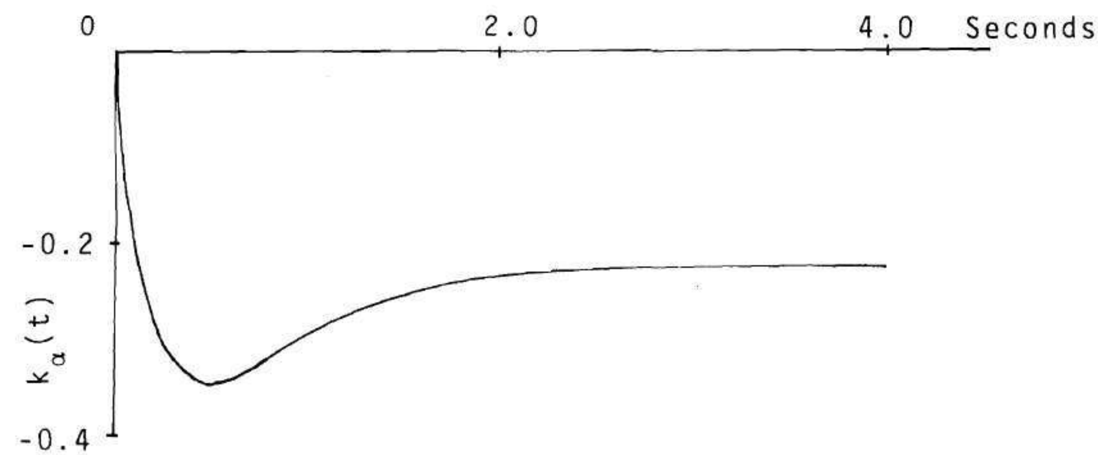

Figure 9. Sensitivity of Kalman Gain to α, Example 1.

Using (4.15) with $\alpha = \alpha_n$, k(0) and k(4) are calculated and found to check against the results shown in Figure 8. Equation (4.15) reveals that as $t \to \infty$, $k(t) \to (\beta_2 - \alpha)$.

(d) Sensitivity of the Kalman Gain. Using (2.43) and (2.44), $k_\alpha(t)$ is given by

$$\dot{k}_\alpha(t) = -2(\alpha + k(t))k_\alpha(t) - 2k(t), \quad k_\alpha(0) = 0 . \tag{4.17}$$

The numerical solution of (4.17) is shown in Figure 9. Store $k_\alpha(t)$ for later use. An analytical expression for $k_\alpha(t)$ can be obtained by partially differentiating k(t) in (4.15) with respect to α, and the result is given below.

$$k_\alpha(t) = \frac{1}{1-(t)^2}\left[\frac{\alpha-\beta_2}{\beta_2} + 2\left\{(1-2\alpha t) + \frac{2(\beta_2^2 - \alpha^2 - \alpha k_0)}{(k_0+\alpha-\beta_2)(k_0+\alpha+\beta_2)}\right\}\phi(t) - \frac{\alpha+\beta_2}{\beta_2}\phi^2(t)\right] \tag{4.18}$$

$$\phi(t) \triangleq \frac{k_0 + \alpha - \beta_2}{k_0 + \alpha + \beta_2}\, e^{-2\beta_2 t} \tag{4.19}$$

Using (4.18) with $\alpha = \alpha_n$, $k_\alpha(0)$ and k (4) are calculated and they check against the results in Figure 9. It is clear from (4.18) that the steady-state value of $k_\alpha(t)$ is $\left(\alpha - \sqrt{\alpha^2 + \frac{q_w}{q_v}}\right)\Big/\sqrt{\alpha^2 + \frac{q_w}{q_v}} = 0.22540333$. This also checks with the results in Figure 9. It can be shown that, using (4.18), as α becomes either positively or negatively very large $k_\alpha(t)$ will approach zero.

(e) Mean Square Trajectory Sensitivity (MSTS) of the Original LSR, $P_{\zeta\zeta}$. Using (2.44), the equation for the augmented state, $P_{\nu\nu}(t)$, is given by

$$\dot{P}_{\nu\nu} = \Sigma(t)P_{\nu\nu} + P_{\nu\nu}\Sigma^T(t) + \Omega , \qquad (4.20)$$

where

$$\Sigma(t) = \begin{bmatrix} -1 & -c(t) & -1 & -c_\alpha(t) \\ k(t) & -1-c(t)-k(t) & k_\alpha(t) & -1-c_\alpha(t) \\ 0 & 0 & -1 & -c(t) \\ 0 & 0 & k(t) & -1-c(t)-k(t) \end{bmatrix}, \qquad (4.21)$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & .015k_\alpha^2(t) & 0 & .015k_\alpha(t)k(t) \\ 0 & 0 & 0.01 & 0 \\ 0 & .015k_\alpha(t)k(t) & 0 & 0.015k^2(t) \end{bmatrix}. \qquad (4.22)$$

Equation (4.20) is solved numerically for $P_{\nu\nu}(t)$, and the mean square trajectory sensitivity of the original LSR, $P_{\zeta\zeta}$, is the (1,1)-element of $P_{\nu\nu}$. $P_{\zeta\zeta}$ is then plotted in Figure 10. $P_{\zeta\zeta}$ has a peak value of approximately 0.82. $P_{\zeta\zeta}$ indicates a considerable sensitivity of the trajectory to α. A 30% variation of α from α_n would result in a dispersion in $x(t)$ as high as 0.27 or 14.5% of $x_d^*(0.7)$. This may be significant in some applications. This is as expected because α appears as the time constant of the plant dynamics. Also,

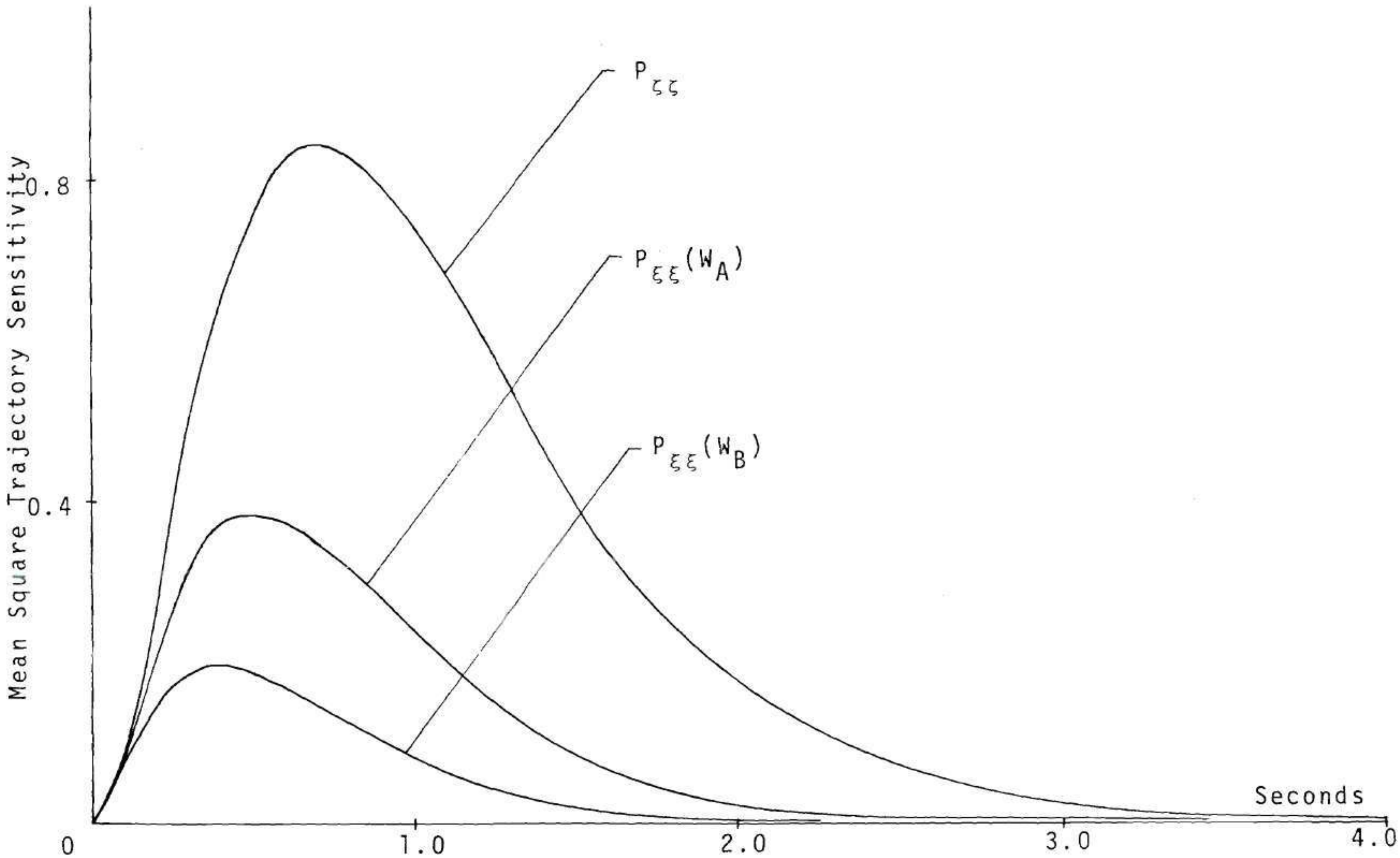

Figure 10. Mean Square Trajectory Sensitivity of the Original LSR and the Compensated System, Example 1.

$k_\alpha(t)$ and $c_\alpha(t)$ are good indicators of the magnitude of the *trajectory sensitivity*. Large values of $k_\alpha(t)$ and $c_\alpha(t)$ are usually associated with a large value of the trajectory sensitivity. This is logical because both $c(t)$ and $k(t)$ are the gains of the LSR; large errors in $c(t)$ and $k(t)$, as suggested by large values $c_\alpha(t)$ and $k_\alpha(t)$, would result in a large deviation of the actual trajectory from the nominal.

Step III. Application of the Compensation Method to Reduce the Trajectory Sensitivity

In the sequel, the compensation method developed in this research is used to effect a reduction in the trajectory sensitivity of the LSR. The main task involved in this method is to obtain the compensating feedback gain, $s(t)$, from the TPBVP defined by (3.60) - (3.64). $c(t)$, $k(t)$, $k_\alpha(t)$, $x_d^*(t)$ and $u_d^*(t)$ are needed in the process of obtaining $s(t)$. The first three have been pre-calculated in Steps II(a), II(c), and II(d), respectively.

(a) Required Time Functions. Using (2.8) and (2.9), $x_d^*(t)$ and $u_d^*(t)$ are given by (4.23) and (4.24) below.

$$\dot{x}_d^* = (-1 - c(t))\underline{x}_d^*, \quad x_d^*(0) = 5, \quad 0 \leq t \leq 4, \tag{4.23}$$

$$u_d^* = -c(t)x_d^*(t) \ . \tag{4.24}$$

$x_d^*(t)$ and $u_d^*(t)$ are then numerically calculated and stored for later use.

(b) *Determination of the Compensating Feedback Gain* S. $s(t)$ must satisfy the necessary conditions *defined* by (3.86) - (3.89). For this particular example, these necessary conditions become

$$\dot{P}_{\theta\theta} = DP_{\theta\theta} + P_{\theta\theta}D^T + \Gamma, \quad P_{\theta\theta}(0) = Q_o; \tag{4.25}$$

$$\dot{\underline{\mu}}_\theta = D\underline{\mu}_\theta + \underline{\bar{\beta}}, \quad \underline{\mu}_\theta(t_o) = [0 \quad 0 \quad 5 \quad 5]^T; \tag{4.26}$$

$$\dot{\Lambda} = -D^T\Lambda - \Lambda D - Q - R, \quad \Lambda(4) = 0 ; \tag{4.27}$$

$$\dot{\underline{\lambda}} = -D^T\underline{\lambda} - 2\Lambda\underline{\bar{\beta}}, \quad \underline{\lambda}(t_f) = \underline{0} ; \tag{4.28}$$

$$\begin{aligned} &2Q_2 sP_{\xi\xi} - 2[(\Lambda_{\xi\xi} + \Lambda^T_{\xi\hat{\xi}})P_{\xi\hat{\xi}} + (\Lambda^T_{\xi x} + \Lambda^T_{\xi\hat{x}})P_{\xi\hat{x}} + (\Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})P_{\hat{\xi}\hat{\xi}} \\ &\quad + (\Lambda^T_{\hat{\xi}x} + \Lambda^T_{\hat{\xi}\hat{x}})P_{\hat{\xi}\hat{x}} + (\Lambda_{\xi x} + \Lambda_{\hat{\xi}x})P^T_{\hat{\xi}x} + (\Lambda_{xx} + \Lambda^T_{x\hat{x}})P_{x\hat{x}} \\ &\quad + (\Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})P^T_{\hat{\xi}\hat{x}} + (\Lambda_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}})P_{\hat{x}\hat{x}}] + 2[(\Lambda^T_{\xi x} + \Lambda^T_{\xi\hat{x}} + \Lambda^T_{\hat{\xi}x} + \Lambda^T_{\hat{\xi}\hat{x}})\mu_\xi \\ &\quad + (\Lambda_{xx} + \Lambda^T_{x\hat{x}} + \Lambda_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}})x^*_d]x^*_d - \mu_\xi(\lambda_\xi + \lambda_{\hat{\xi}}) = 0 , \end{aligned} \tag{4.29}$$

where

$$D = \begin{bmatrix} -1 & -s(t) & -1 & 0 \\ k(t) & -1-s(t)-k(t) & k_\alpha(t) & -1-k_\alpha(t) \\ 0 & 0 & -1 & -s(t) \\ 0 & 0 & k(t) & -1-s(t)-k(t) \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 0 & 0 & m_1\mu_\xi & m_1\mu_\xi \\ & 0.015k^2_\alpha(t) & m_1\mu_\xi & m_1\mu_\xi + 0.015k(t)k_\alpha(t) \\ & & 0.01 & 2m_1x^*_d \\ & & & 2m_1x^*_d + 0.015k^2(t) \end{bmatrix},$$

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}, \quad R = \begin{bmatrix} q_2 s^2(t) & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

$$Q_o = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 25.02 & 25 \\ & & & 25 \end{bmatrix}, \quad m_1 = u_d^* + s(t)x_d^*(t).$$

Since P, Q, R and Q_o are symmetric matrices, only their upper triangular elements are shown in the above. It should be noted that the results of Properties 3.1 and 3.2 (i.e., $\mu_x(t) = \mu_{\hat{x}}(t)$ and $\mu_\xi(t) = \mu_{\hat{\xi}}(t)$, respectively) and the combined results of Properties 3.3 and 2.1 (i.e., $\mu_x(t) = x_d^*(t)$) have been used in (4.27) and (4.29). Since $x_d^*(t)$ has already been calculated in (a), only $\mu_\xi(t)$ is needed to be calculated for use in (4.27) and (4.29). After expanding (3.76) in terms of the sub-vectors and using the relationships: $\mu_x = \mu_{\hat{x}}$, $\mu_\xi = \mu_{\hat{\xi}}$ and $\mu_x = x_d^*$, the differential equation for μ_ξ is given by

$$\dot{\mu}_\xi = -(1 + s(t))\mu_\xi - x_d^*(t), \quad \mu_\xi(0) = 0 . \qquad (4.30)$$

Equations (4.25) - (4.27) which represent a TPBVP are then numerically solved using the gradient algorithm for two sets of values of the weights q_1 and q_2, see (3.55), i.e.,

$$W_A \triangleq \{q_1, q_2 \mid q_1 = 5, \quad q_2 = 1\} \;, \tag{4.31}$$

$$W_B \triangleq \{q_1, q_2 \mid q_1 = 10, \; q_2 = 1\} \;. \tag{4.32}$$

For the sake of convenience, the weight set W_A or W_B will be written as an argument of s, J, P , and so on, to indicate the correspondence of these terms with W_A or W_B, for example,

$s(W_A)$ = the optimal compensating gain s(t) corresponding to the weight set W_A,

$P_{\xi\xi}(W_A)$ = the MSTS of the compensated system corresponding to the weight set W_A,

$J(W_A)$ = the overall cost corresponding to the weight set W_A.

$s(W_A)$ and $s(W_B)$ are plotted in Figure 11. $P_{\xi\xi}(W_A)$ and $P_{\xi\xi}(W_B)$ are plotted in Figure 10 together with the MSTS of the original LSR, $P_{\xi\xi}$, for the sake of comparison. Referring to Figure 10, $P_{\xi\xi}(W_A)$ shows a reduction in the MSTS from $P_{\zeta\zeta}$ of as high as 54%, and $P_{\xi\xi}(W_B)$ as high as 77%. In order to compare the overall cost J, see (4.1), of the original LSR and that of the compensated system under the nominal conditions, J is calculated for each individual case by means of 50 Monte Carlo simulations. The results are given below.

$$J_o = 5.2010571, \; J(W_A) = 5.2070858, \; J(W_B) = 5.2123299,$$

where J_o is the overall cost of the original LSR. It is clear from these figures that the compensated system causes a very

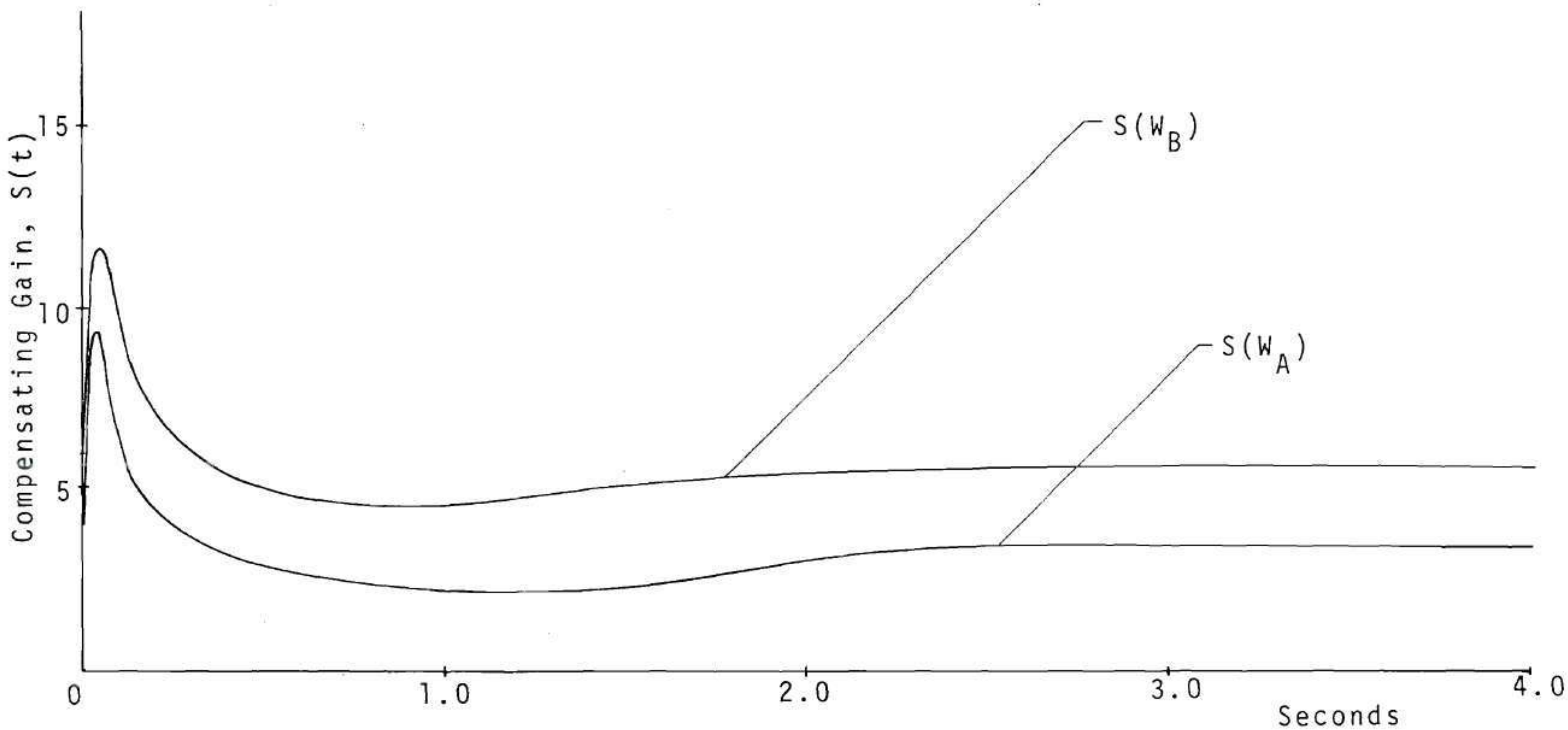

Figure 11. Compensating Gains for Weight Sets W_A and W_B, Example 1.

small increase in the overall cost ($J(W_A)$ represents 0.12% increase and $J(W_B)$ 0.22%). This is highly desirable because a small overall cost J is one of the main objectives of an LSR. Now, consider $s(W_A)$ and $s(W_B)$ in Figure 11. Both $s(W_A)$ and $s(W_B)$ appear to become constant as t approaches the final time. This suggests the existence of the steady-state values of $s(W_A)$ and $s(W_B)$, in which case the constant values of $s(W_A)$ and $s(W_B)$ can be used just as in the case of a steady-state Kalman filter. Now compare the dispersions of x, Δx, for 30% variation in α from its nominal value. The approximate results are given below.

Original LSR: $\Delta x(0.7) = .27$ or 14.5% of $x_d^*(0.7)$,

Compensated system with $s(W_A)$: $\Delta x(0.7) = .178$ or 9.55% of $x_d^*(.7)$,

Compensated system $s(W_B)$: $\Delta x(0.7) = .114$ or 6.13% of $x_d^*(.7)$.

The dispersion $\Delta x(t)$ can be further decreased by using a larger value of weight q_1.

As it has been pointed out in Chapter III, a proper set $\{q_1, q_2\}$ cannot be determined analytically. W_A and W_B had to be varied by trial and error until the desired magnitude of the trajectory sensitivity of interest is obtained. $s(W_A)$ and the corresponding $P_{\xi\xi}(W_A)$ were obtained in the sixth iteration, and $s(W_B)$ and $P_{\xi\xi}(W_B)$ in the seventh iteration of the gradient algorithm. Each iteration took approximately 13.4 seconds of the CPU time.

Example 2

In this example, the nominal LSR to be considered is the same as that in Example 1 except now the initial value of the error covariance, p_0, is subject to modelling errors. In some cases of pure estimation applications, either p_0 or the error covariance itself, $p(t)$, $t > t_0$, is adjusted to improve the poor performance, caused by modelling errors, of the Kalman filter. Therefore, it is interesting to consider the trajectory sensitivity of the LSR to modelling errors in p_0. For clarity, the LSR problem to be considered is defined again in the following.

$$\min_{u(t)} J = \min_{u(t)} \frac{1}{2} E\{\int_0^4 [x^2(t) + u^2(t)]dt\}, \tag{4.33}$$

subject to

$$\dot{x} = -x + u + w, \quad E\{x(t_0)\} = 5, \; var\{x(t_0)\} = \alpha ; \tag{4.34}$$

$$z = x + v,$$

where α is the parameter whose nominal value is $\alpha_n = 0.02$, and all others remain the same as in Example 1.

Step I: Identifications of Variables

Now $\alpha_n = 0.02$ and $p_0 = 0.02$. Everything else remains the same as in Step I of Example 1.

<u>Step II</u>: Sensitivity Analysis of the Original LSR

(a) The Deterministic Controller. The controller gain, c(t), is the same as that of Example 1.

(b) Sensitivity of the Controller Gain. Since c(t) is independent of α, one has

$$c_{\alpha}(t) \equiv 0$$

(c) The Kalman Filter. The Kalman gain, k(t), is the same as that of Example 1 under the nominal condition.

(d) Sensitivity of the Kalman Gain. Using (2.33) and (2.34), $k_{\alpha}(t)$ is given by

$$\dot{k}_{\alpha} = -2(1 - k(t))k_{\alpha}, \ k(0) = k_{o}, \tag{4.36}$$

where

$$k_{o} \triangleq \frac{1}{q_v} \frac{\partial p_o}{\partial \alpha}\bigg|_{\alpha=\alpha_n} = \frac{1}{q_v} = \frac{1}{0.015}$$

The numerical solution of (4.36) is shown in Figure 12. Store $k_{\alpha}(t)$ for later use. Initially $k_{\alpha}(t)$ has a value of approximately 66.7 and decays exponentially as t increases. Analytical solution for k (t) can be obtained by partially differentiating k(t) with respect to α. The expression for k(t) can be obtained from (4.15) with α replaced by 1 and k_o by α/q_v, the result is given below.

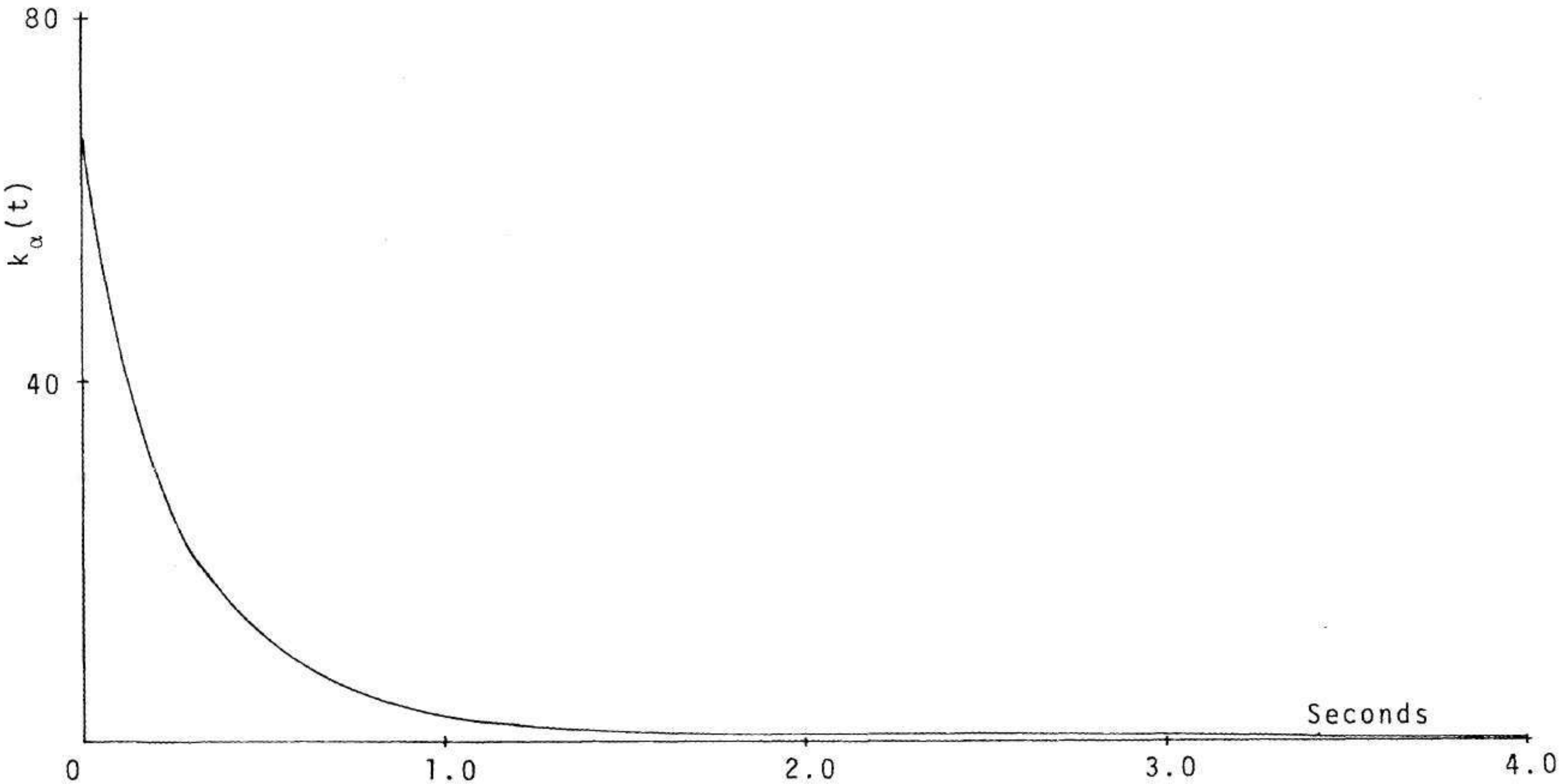

Figure 12. Sensitivity of Kalman Gain to α, Example 2.

$$k(t) = \frac{(\beta_2 - 1) + (\beta_2 + 1)\dfrac{\alpha/q_v + 1 - \beta_2}{\alpha/q_v + 1 + \beta_2} e^{-2\beta_2 t}}{1 - \dfrac{\alpha/q_v + 1 - \beta_2}{\alpha/q_v + 1 + \beta_2} e^{-2\beta_2 t}} , \qquad (4.37)$$

where

$$\beta_2 = \sqrt{1 + \frac{q_w}{q_v}} .$$

Hence, $k_\alpha(t)$ is given by

$$k_\alpha(t) = \frac{4\beta_2^2 \, e^{-2\beta_2 t}}{q_v(\frac{\alpha}{q_v} + 1 + \beta_2)^2} \Bigg/ \left[1 - \frac{\alpha/q_v + 1 - \beta_2}{\alpha/q_v + 1 + \beta_2} e^{-2\beta_2 t}\right]^2 . \qquad (4.38)$$

According to (4.38), $k_\alpha(t)$ exhibits an exponential type of decay with the initial value of $\frac{1}{q_v}$, and this appears to be in good agreement with the results in Figure 12.

(e) The MSTS of the Original LSR, $P_{\zeta\zeta}$. Using (2.54), the equation for $P_{\nu\nu}(t)$ is given by

$$\dot{P}_{\nu\nu} = \Sigma(t)P_{\nu\nu} + P_{\nu\nu}\Sigma^T(t) + \Omega, \qquad (4.39)$$

where

$$\Sigma(t) = \begin{bmatrix} -1 & -c(t) & 0 & 0 \\ k(t) & -1-c(t)-k(t) & k_\alpha(t) & 0 \\ 0 & 0 & -1 & -c(t) \\ 0 & 0 & k(t) & -1-c(t)-k(t) \end{bmatrix}, \qquad (4.40)$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & .015k_{\alpha}^{2}(t) & 0 & .015k(t)k_{\alpha}(t) \\ 0 & 0 & .01 & 0 \\ 0 & .015k(t)k_{\alpha}(t) & 0 & .015k^{2}(t) \end{bmatrix}, \quad (4.41)$$

Equation (4.39) is numerically solved for $P_{\nu\nu}(t)$. And the MSTS of the original LSR, $P_{\zeta\zeta}(t)$, is then plotted in Figure 13. It has a peak value of approximately 0.076 at t = 0.85 second and gradually decrease to a very small value as t approaches 4 seconds. In this example, only the Kalman filter is subject to modelling errors whereas the deterministic controller is not. In this situation it is interesting to see the behavior of the mean square estimate sensitivity (MSES), $P_{\hat{\zeta}\hat{\zeta}}$; however, it should be emphasized that $P_{\hat{\zeta}\hat{\zeta}}$ is not of the prime concern in this research. $P_{\hat{\zeta}\hat{\zeta}}$ is plotted in Figure 14. The peak value of $P_{\hat{\zeta}\hat{\zeta}}$ is approximately 53 times that of $P_{\xi\xi}$ in Figure 13. This corresponds to a fairly high peak value of $K_{\alpha}(t)$, see Figure 12. However, $P_{\hat{\zeta}\hat{\zeta}}$ decays to a small value faster than $P_{\xi\xi}$; this corresponds to the exponentially fast decay of $K_{\alpha}(t)$.

Step III: Application of the Compensation Method to Reduce the Trajectory Sensitivity.

In this step, the compensation method will be applied to the LSR under present consideration to reduce the trajectory sensitivity due to α.

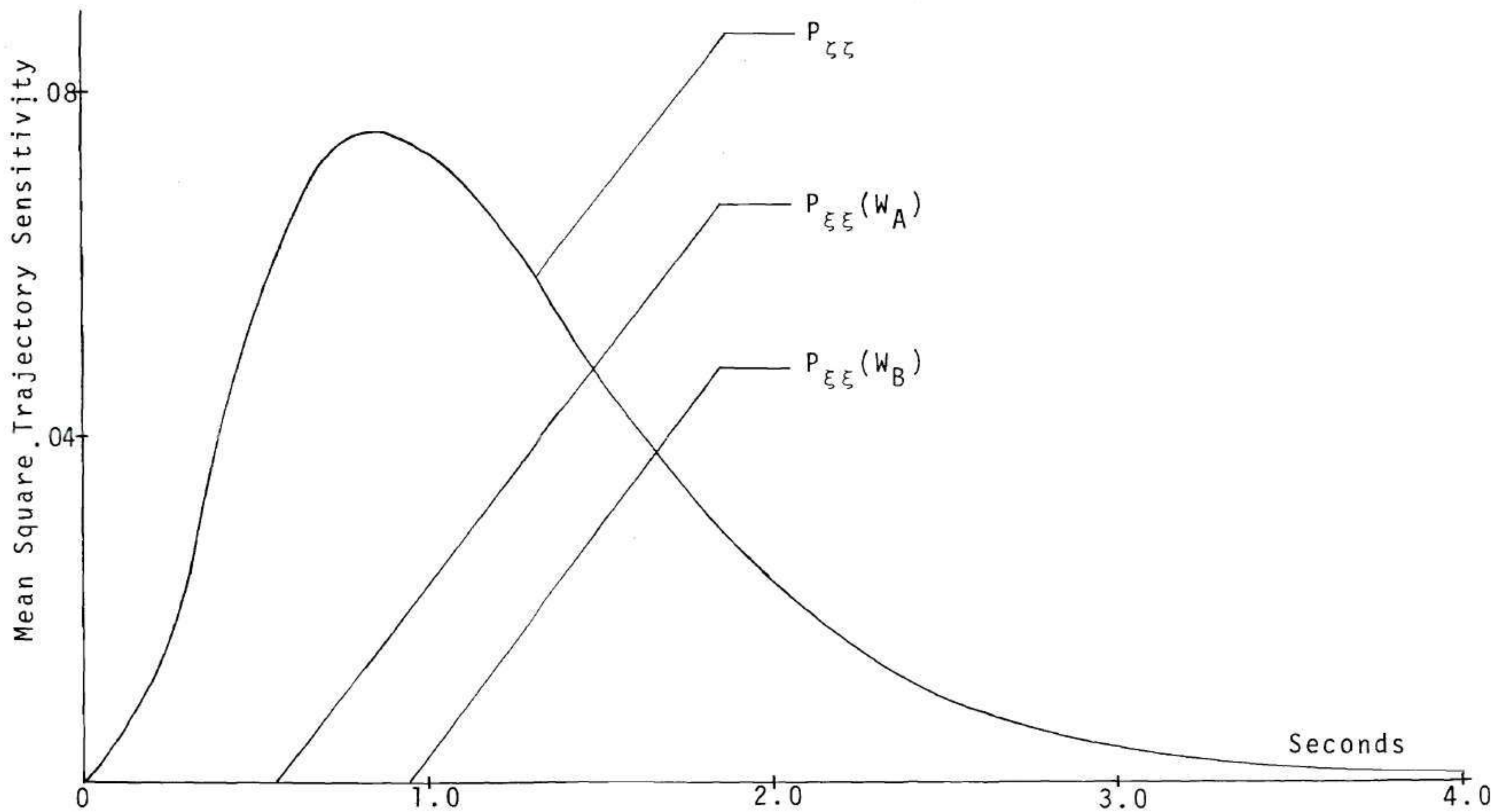

Figure 13. Mean Square Trajectory Sensitivity of the Original and the Compensated System, Example 2.

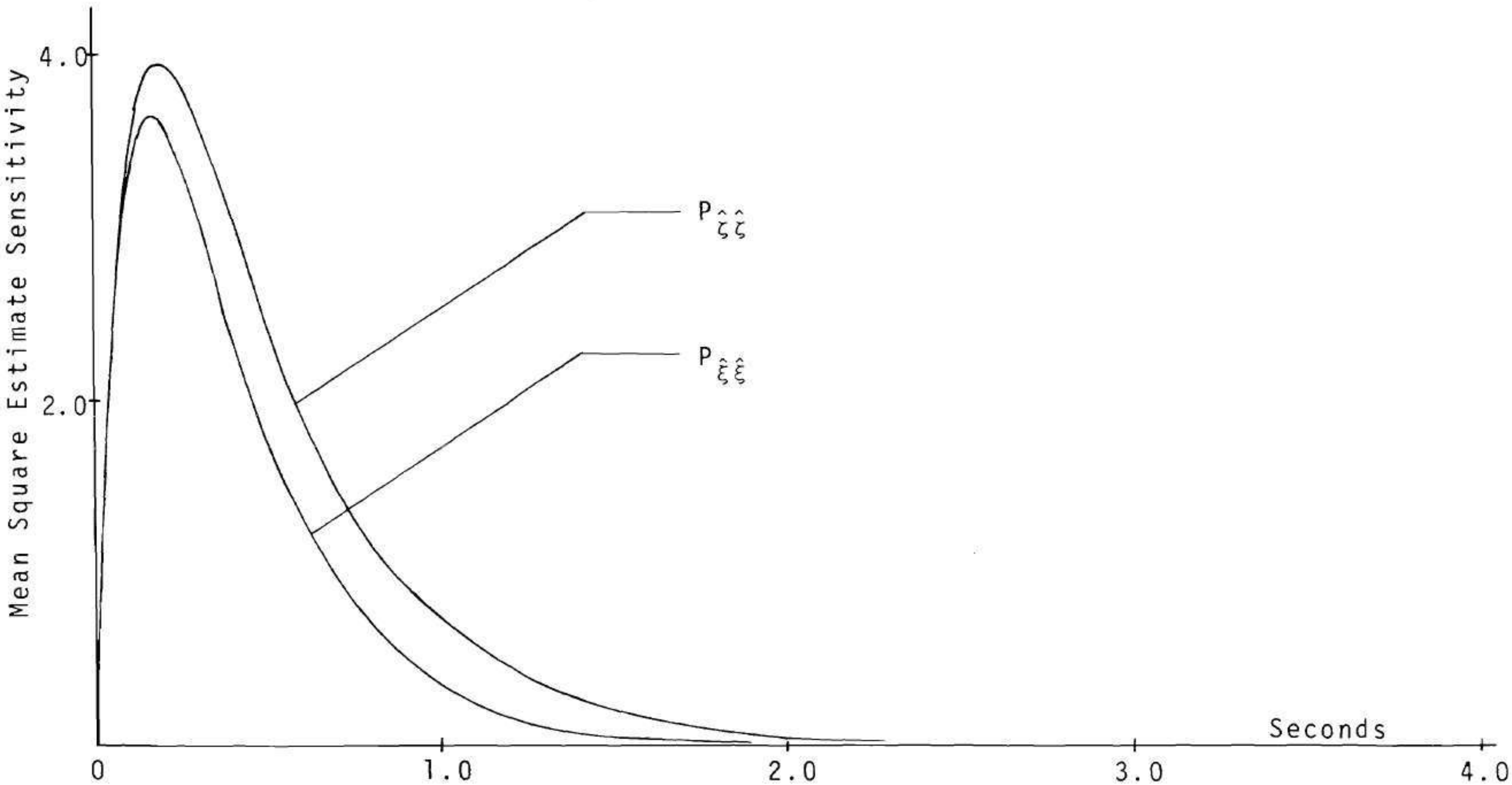

Figure 14. Mean Square Estimate Sensitivity of the Original LSR and the Compensated System, Example 2.

(a) Required Time Functions. The required time functions are $c(t)$, $k(t)$, $k_\alpha(t)$, $x_d^*(t)$ and $u_d^*(t)$. $k_\alpha(t)$ has been generated in Step II(d), and the others have been generated previously in Example 1.

(b) Determination of the Compensating Feedback Gain S. The equations representing the necessary conditions are the same as those in Example 1, i.e., equations (4.25) - (4.29). But now the matrix D is given by

$$D = \begin{bmatrix} -1 & -s(t) & 0 & 0 \\ k(t) & -1-s(t)-k(t) & k_\alpha(t) & -k_\alpha(t) \\ 0 & 0 & -1 & -s(t) \\ 0 & 0 & k(t) & -1-s(t)-k(t) \end{bmatrix} .$$

The expressions for Γ, Q, R, Q_0 and m_1 remains the same. The D.E. for $\mu_\xi(t)$ (see 4.30) now becomes

$$\dot{\mu}_\xi = -(1+s(t))\mu_\xi \,, \quad \mu_\xi(0) = 0 \,. \tag{4.42}$$

The solution of (4.42) is

$$\mu_\xi(t) = 0 \quad \text{for all } t. \tag{4.43}$$

This identically zero value of $\mu_\xi(t)$ is to be used in (4.27) and in the expression for Γ.

Equations (4.25) - (4.27) are numerically solved, using the gradient algorithm, for two sets of values of weights, q_1 and q_2:

$$W_A = \{q_1, q_2 | q_1 = .01,\ q_2 = 10^5\}, \qquad (4.44)$$

$$W_B = \{q_1, q_2 | q_1 = 1,\ q_2 = 10^3\}\ . \qquad (4.45)$$

Again, W_A and W_B have to be varied by trial and error until the desired magnitude of the trajectory sensitivity of interest is obtained. The resulting optimal compensating gains, $s(W_A)$ and $s(W_B)$, are plotted together in Figure 15. The corresponding $P_{\xi\xi}(W_A)$ and $P_{\xi\xi}(W_B)$ are plotted in Figure 13 together with $P_{\zeta\zeta}$ for comparison. Both $P_{\xi\xi}(W_A)$ and $P_{\xi\xi}(W_B)$ are very small. They are of the order of magnitude of 10^{-5}. $P_{\xi\xi}(W_A)$ is slightly greater than $P_{\xi\xi}(W_B)$. The MSES corresponding to W_B of the compensated system, denoted by $P_{\hat{\xi}\hat{\xi}}(W_B)$, is plotted in Figure 14 together with $P_{\hat{\zeta}\hat{\zeta}}$. The following is very interesting to note. In Figure 13 $P_{\xi\xi}(W_B)$ shows a dramatic reduction in the trajectory sensitivity from $P_{\zeta\zeta}$, whereas in Figure 14 $P_{\hat{\xi}\hat{\xi}}(W_B)$ shows a small increase from $P_{\hat{\xi}\hat{\xi}}$. This strongly indicates a trade-off between the MSTS and the MSES, which is also supported by the result that $P_{\xi\xi}(W_B)$ shows a small decrease from $P_{\xi\xi}(W_A)$ while $P_{\hat{\xi}\hat{\xi}}(W_B)$ shows a noticeable increase from $P_{\hat{\xi}\hat{\xi}}(W_A)$. (This result is not shown in Figure 14 because $P_{\hat{\xi}\hat{\xi}}(W_A)$ and $P_{\hat{\xi}\hat{\xi}}(W_B)$ are too close together to be shown as two distinct curves for the scales chosen for the figure.) Under this circumstance, an attempt to reduce the MSES will not result in a corresponding reduction in the MSTS; in fact, it will result in the contrary. This indicates the possible failure of Stavroulakis and Sarachik's compensation method [50] which

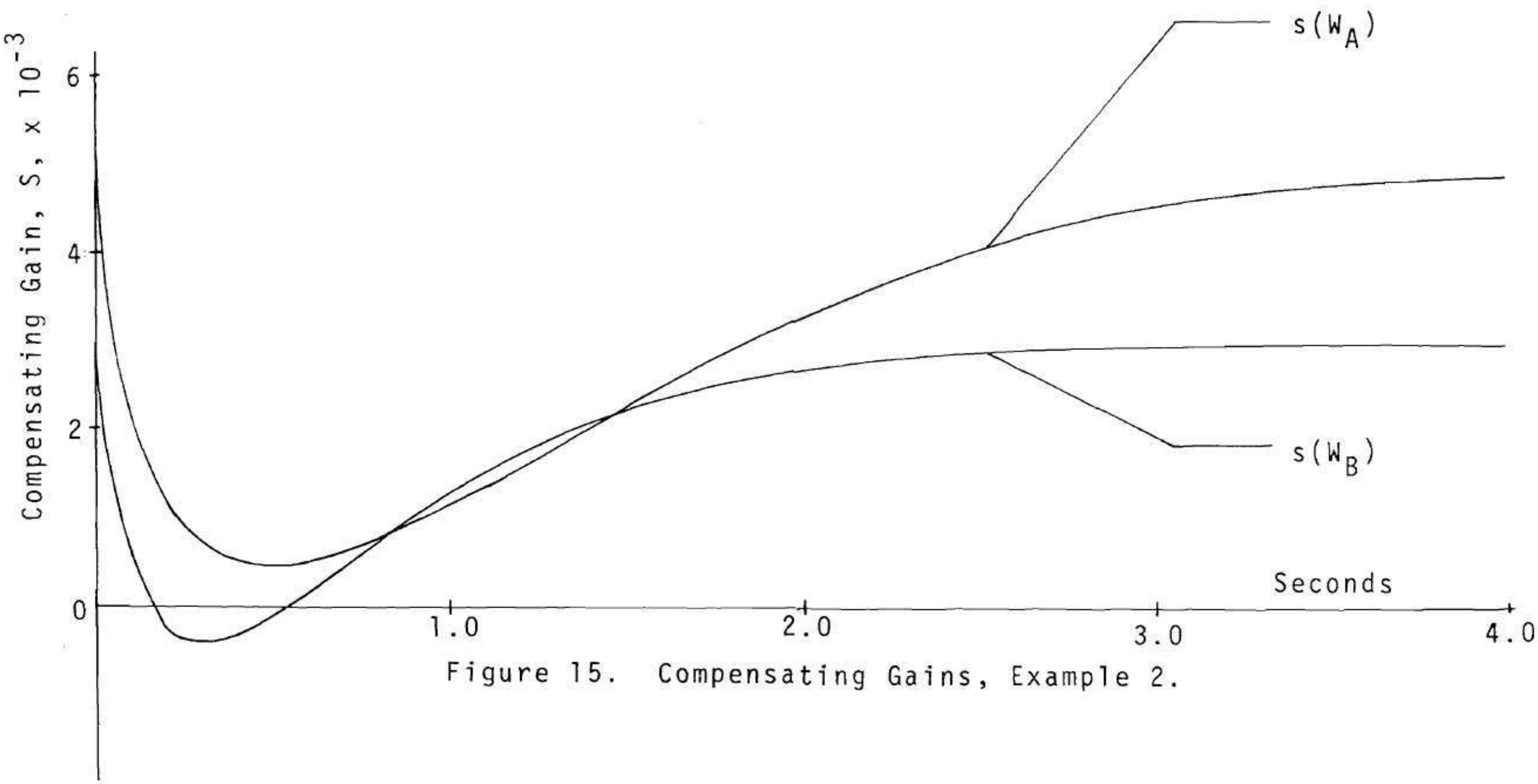

Figure 15. Compensating Gains, Example 2.

is based on the hope that the attempted minimization of the MSES will result in a corresponding reduction in the MSTS.

In order to compare the overall cost, J, incurred by the use of the compensation method with that of the original LSR, J's are computed for the nominal conditions and shown below.

$$J_o = 5.2010571, \; J(W_A) = 5.2010598, \; J(W_B) = 5.2010607, \qquad (4.46)$$

where these notations have the same meanings as in Example 1. It is clear that both $J(W_A)$ and $J(W_B)$ show an extremely small increase from J_o. As pointed out in Example 1, this is highly desirable. The approximate figures for Δx of the original LSR and the compensated system are given below for comparison for 30% variation in α from its nominal value.

Original LSR : $\Delta x(.85) = .0824$ or 5.5% of $x_d^*(.85)$,
Compensated system with $s(W_A)$: $\Delta x(.85)=.0002$ or .01% of $x_d^*(.85)$,
Compensated system with $s(W_B)$: $\Delta x(.85)=.00003$ or .002% of $x_d^*(.85)$.

The amount of the CPU time and the computation involved in the solution of the TPBVP is approximately the same as that of Example 1. Finally, $s(W_A)$ and $s(W_B)$ of Figure 15 suggest the existence of their steady-state values as t approaches 4 seconds.

Example 3

A typical process control system known as "a stirred tank" is considered in this example [58, p. 7]. Consider the stirred tank of Figure 16. The tank is fed with two feeds of dissolved materials with concentrations c_1 and c_2, at the flow rates of $F_1(t)$ and $F_2(t)$, respectively. It is assumed that the tank is stirred well by a propeller so that the concentration of the outgoing flow equals the concentration $c(t)$ in the tank. Consider a steady-state situation where all quantities are constant, say F_{10}, F_{20}, F_{30} for the flow rates, V_0 for the volume, and c_0 for the concentration in the tank. Now assume that the system is perturbed by a small deviation from the steady state conditions.

$$F_1(t) = F_{10} + u_1(t), \tag{4.47}$$

$$F_2(t) = F_{20} + u_2(t), \tag{4.48}$$

$$V(t) = V_0 + x_1(t) , \tag{4.49}$$

$$c(t) = c_0 + x_2(t), \tag{4.50}$$

where $u_1(t)$ and $u_2(t)$ are the control variables, and $x_1(t)$ and $x_2(t)$ are the state variables. It can be shown these four quantities satisfy the following differential equation.

$$\dot{\underline{x}}(t) = F\underline{x}(t) - G\underline{u}(t) + \underline{w} , \tag{4.51}$$

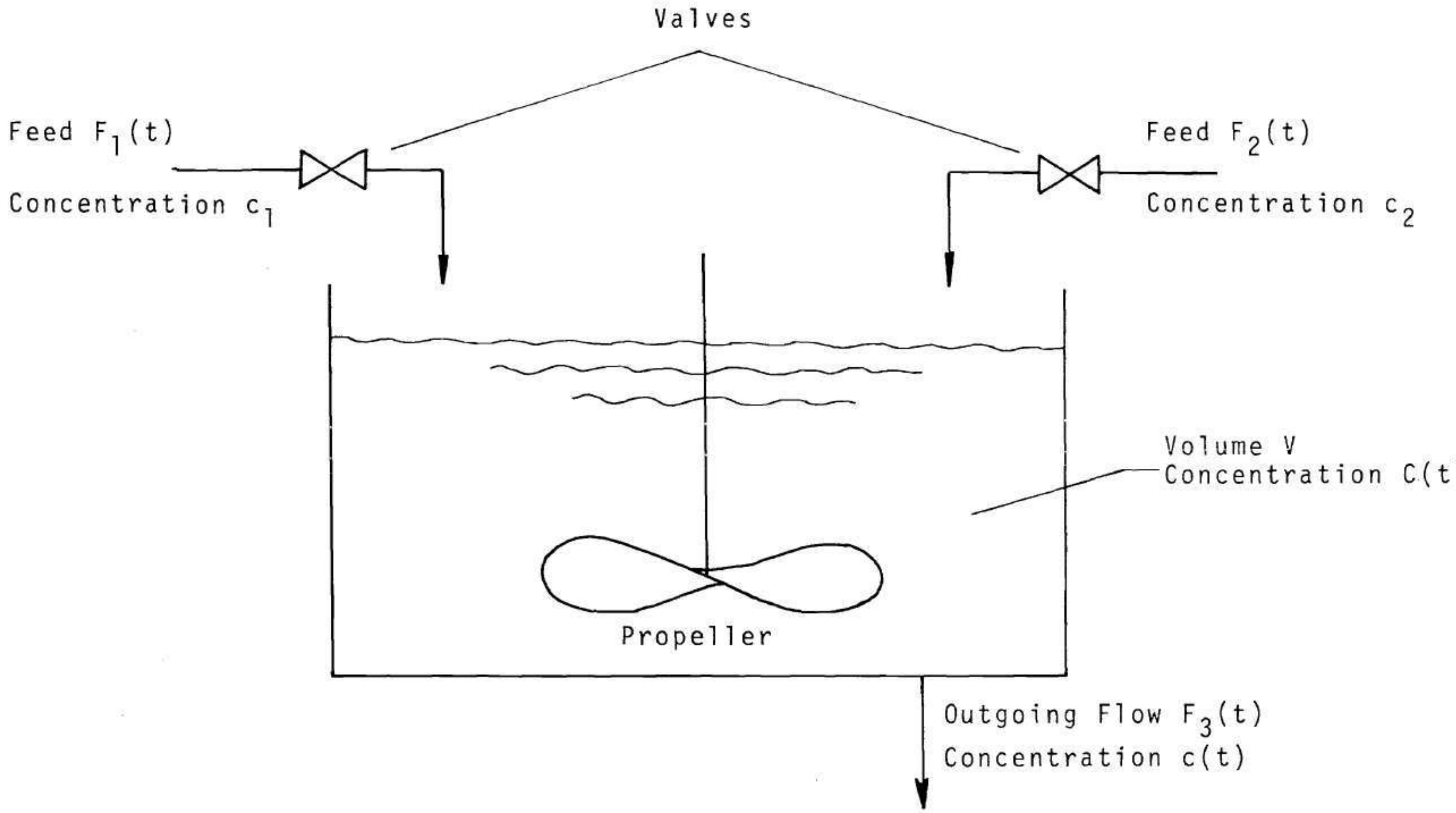

Figure 16. A Stirred Tank Process, Example 3.

$$z = H\underline{x} + v \ , \tag{4.52}$$

where

$$\underline{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} ,$$

$$\underline{u}(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} ,$$

$$F \triangleq \begin{bmatrix} \frac{1}{-2\theta} & 0 \\ 0 & \frac{1}{-\theta} \end{bmatrix},$$

$\theta \triangleq \frac{V_o}{F_{30}}$ (known as hold up time of tank),

$$G \triangleq \begin{bmatrix} 1 & 1 \\ \frac{c_1 - c_o}{V_o} & \frac{c_2 - c_o}{V_o} \end{bmatrix},$$

H is a measurement 1 x 2 matrix, z is a scalar measurement, $\underline{w}$ and v are additive white Gaussian noises with the following properties:

$$E\{\underline{w}(t)\} = E\{\underline{v}(t)\} = \underline{0} \ ,$$

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = Q_w(t)\delta(t-\tau),$$

$$E\{\underline{w}(t)\underline{v}(\tau)\} = 0 \ ,$$

$$E\{v(t)v(t-\tau)\} = q_v(t)\delta(t-\tau).$$

The LSR problem for the process control system of Figure 16 can be stated as follows.

$$\min_{u(t)} J = \min_{\underline{u}(t)} \frac{1}{2} E\{\int_{t_o}^{t_f} [||\underline{x}(t)||_A + ||\underline{u}(t)||_B^2]dt\} \qquad (4.53)$$

subject to (4.51) and (4.52), where A(t) is a symmetric positive semi-definite 2 x 2 matrix and B(t) is a symmetric positive definite 2 x 2 matrix. This simply means that the incremental feed rates $u_1(t)$ and $u_2(t)$ of feeds $F_1(t)$ and $F_2(t)$, respectively, are to be found such that the volume, V(t), and the concentration, c(t), of the fluid in the tank remain close to their steady-state values, V_o and c_o, respectively. The second term of the integrand of (4.53) helps to ensure that $u_1(t)$ and $u_2(t)$ remain finite for non-zero B. Using the following numerical values,

$$F_{10} = 0.015 \text{ m}^3/\text{sec.},$$

$$F_{20} = 0.005 \text{ m}^3/\text{sec.},$$

$$F_{30} = 0.02 \text{ m}^3/\text{sec.},$$

$$c_o = 1.25 \text{ kmol/m}^3,$$

$$V_o = 0.5 \text{ m}^3,$$

equation (4.51) and (4.52) now become

$$\underline{\dot{x}} = \begin{bmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 1 \\ \frac{c_1 - 1.25}{0.5} & \frac{c_2 - 1.25}{0.5} \end{bmatrix} \underline{u} + \underline{w} , \quad (4.54)$$

$$z = [1 \quad 1] \; \underline{x} + v . \quad (4.55)$$

Also, the following numerical values will be used,

$$A = \begin{bmatrix} .005 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} .033 & 0 \\ 0 & .3 \end{bmatrix}, \quad (4.56)$$

$$Q_w = \begin{bmatrix} 0 & 0 \\ 0 & .0006 \end{bmatrix}, \quad q_v = .0005. \quad (4.57)$$

Now, the final form of the LSR problem for the process control system is given by (4.53), (4.54) and (4.55). The trajectory sensitivity of the system due to parameter variations in θ, c_1 and c_2 will be considered. Let the nominal values of θ, c_1 and c_2 be 25 seconds, 1 kmol/m^3, and 2 kmol/m^3, respectively.

Step I: Identifications of Variables.

For convenience the notations analogous to (3.88)-(3.92) are

$$\underline{\zeta}^i(t) \triangleq \left.\frac{\partial \underline{x}}{\partial \alpha_i}\right|_{\underline{\alpha}=\underline{\alpha}_n}, \quad \underline{\zeta}^i(t) = [\zeta_1^i \; \zeta_2^i \ldots \; \zeta_n^i]^T, \quad (4.58)$$

$$\hat{\underline{\zeta}}^i \triangleq \left.\frac{\partial \underline{x}}{\partial \alpha_i}\right|_{\underline{\alpha}=\underline{\alpha}_n}, \quad \hat{\underline{\zeta}}^i = [\hat{\zeta}_1^i \; \hat{\zeta}_2^i \ldots \; \hat{\zeta}_n^i]^T , \quad (4.59)$$

$$\underline{\nu}^i \triangleq [\underline{\zeta}^{i^T} \quad \hat{\underline{\zeta}}^{i^T} \quad \underline{x}^T \ \hat{\underline{x}}^T]^T, \tag{4.60}$$

$$P^i_{\nu\nu}(t) \triangleq E\{\underline{\nu}^i(t)\underline{\nu}^{i^T}(t)\} \ , \tag{4.61}$$

$$P^i_{xy}(t) \triangleq E\{\underline{x}^i(t)\underline{y}^{i^T}(t)\} \text{ for arbitrary } \underline{x}(t) \text{ and } \underline{y}(t), \tag{4.62}$$

$$\Sigma^i(t) \triangleq \begin{bmatrix} F & -GC & F_{\alpha_i} & -GC_{\alpha_i} - G_{\alpha_i}C \\ KH & F-GC-KH & K_{\alpha_i}H & F_{\alpha_i} - GC_{\alpha_i} - G_{\alpha_i}C \\ 0 & 0 & F & -GC \\ 0 & 0 & KH & F-GC-KH \end{bmatrix}, \tag{4.63}$$

$$\Omega^i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_{\alpha_i}Q_vK^T_{\alpha_i} & 0 & K_{\alpha_i}Q_vK^T \\ 0 & 0 & Q_w & 0 \\ 0 & KQ_vK^T_{\alpha_i} & 0 & KQ_vK^T \end{bmatrix}, \tag{4.64}$$

where the superscript i denotes the correspondence with the parameter component α_i, and the subscript α_i denotes the partial differentiation with respect to α_i evaluated at $\underline{\alpha} = \underline{\alpha}_n$.

With reference to (2.12)-(2.19) and (2.23), the following identifications are made.

$$\underline{\alpha} = [\alpha_1 \ \alpha_2 \ \alpha_3]^T \triangleq [c_1 \quad c_2 \ \theta]^T \ ,$$

$$\underline{\alpha}_n = [1 \quad 2 \quad 25]^T \ ,$$

$$F = \begin{bmatrix} -\dfrac{1}{2\alpha_3} & 0 \\ 0 & -\dfrac{1}{\alpha_3} \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ \dfrac{\alpha_1 - 1.25}{0.5} & \dfrac{\alpha_2 - 1.25}{0.5} \end{bmatrix},$$

$$\underline{x}_o = \begin{bmatrix} 0 & .1 \end{bmatrix}^T, \quad P_o = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 3\times10^{-4} \end{bmatrix},$$

$$t_o = 0, \quad t_f = 7.5 \text{ secs},$$

Step size of numerical integration = 0.05 sec.

A, B, Q_w and q_v have already been identified in (4.56) and (4.57).

<u>Step II</u>: Sensitivity Analysis of the Original LSR

(a) The deterministic Controller. From (2.10) and (2.11), the deterministic controller gain matrix, C(t), is given by

$$C(t) = \begin{bmatrix} .033 & 0 \\ 0 & .3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -.5 & 1.5 \end{bmatrix}^T S_2, \qquad (4.65)$$

$$\dot{S}_2 = -S_2 \begin{bmatrix} -.02 & 0 \\ 0 & -.04 \end{bmatrix} - \begin{bmatrix} -.02 & 0 \\ 0 & -.04 \end{bmatrix}^T S_2 + C^T(t) \begin{bmatrix} .033 & 0 \\ 0 & .3 \end{bmatrix} C(t) - \begin{bmatrix} .005 & 0 \\ 0 & .2 \end{bmatrix},$$

$$S_2(t_f) = 0 \qquad (4.66)$$

Equation (4.65) and (4.66) are numerically solved for C(t) which is then shown in Figure 17 with the numbers in the parentheses to indicate the positions of the elements of C(t). C(t) remains nearly constant for $0 \leq t \leq 5$.

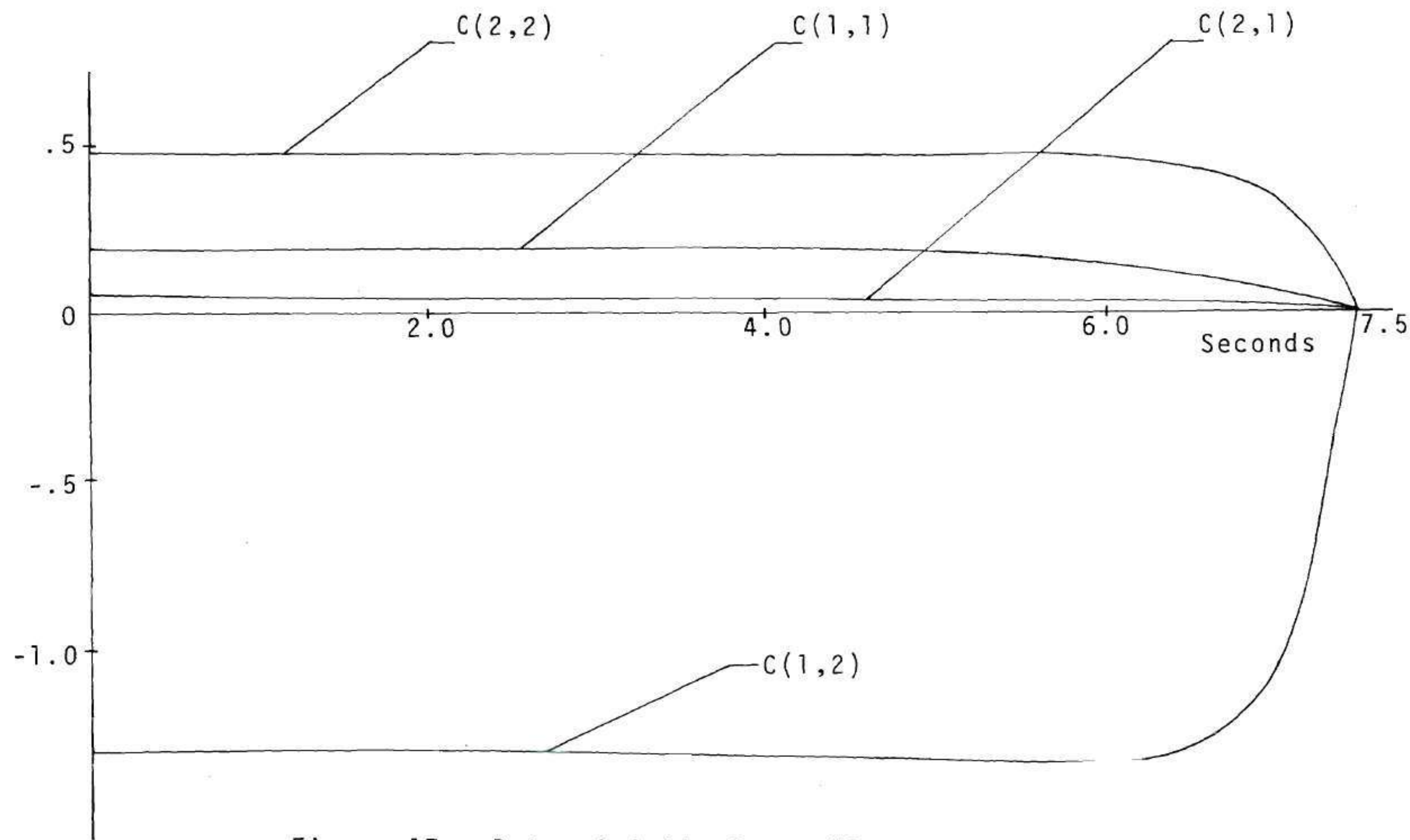

Figure 17. Deterministic Controller Gain, Example 3.

(b) Sensitivity of the Controller Gain. Using (2.41) and (2.42), the controller gain sensitivity, C_{α_i}, i = 1,2,3, is given by

$$C_{\alpha_i} = B^{-1}[G^T_{\alpha_i}S_2 + G^T S_{2\alpha_i}], \; i = 1,2,3; \qquad (4.67)$$

$$\dot{S}_{2\alpha_i} = -S_{2\alpha_i}[F - GB^{-1}G^T S_2] - [F - GB^{-1}G^T S_2]^T S_{2\alpha_i} - [S_2 F_{\alpha_i} + F^T_{\alpha_i} S_2]$$

$$+ S_2[G_{\alpha_i}B^{-1}G^T + GB^{-1}G^T_{\alpha_i}]S_2, \; S_{2\alpha_i}(t_f) = 0, \; i = 1,2,3, \qquad (4.68)$$

where

$$F_{\alpha_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; F_{\alpha_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; F_{\alpha_3} = \begin{bmatrix} .0008 & 0 \\ 0 & .0016 \end{bmatrix}, \qquad (4.69)$$

$$G_{\alpha_1} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \; G_{\alpha_2} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \; G_{\alpha_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad (4.70)$$

Equations (4.67) and (4.68) are solved numerically for C_{α_i}, i = 1,2,3. Then, C_{α_i}, i = 1,2,3, are plotted in Figures 18, 19 and 20, respectively. It appears from these Figures that in order of magnitude, C_{α_1} is the largest, then C_{α_2}, and C_{α_3}.

(c) The Kalman Filter. Using (2.28) and (2.29), the Kalman filter gain matrix, K(t), is given by

$$K(t) = P\begin{bmatrix} 1 & 1 \end{bmatrix}^T (.0005)^{-1}, \qquad (4.71)$$

$$\dot{P} = \begin{bmatrix} -.02 & 0 \\ 0 & -.04 \end{bmatrix} P + P \begin{bmatrix} -.02 & 0 \\ 0 & -.04 \end{bmatrix}^T - K(t)(.0005)^{-1} K^T(t) + \begin{bmatrix} 0 & 0 \\ 0 & .0006 \end{bmatrix},$$

$$P(t_o) = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 3\times10^{-4} \end{bmatrix}. \qquad (4.72)$$

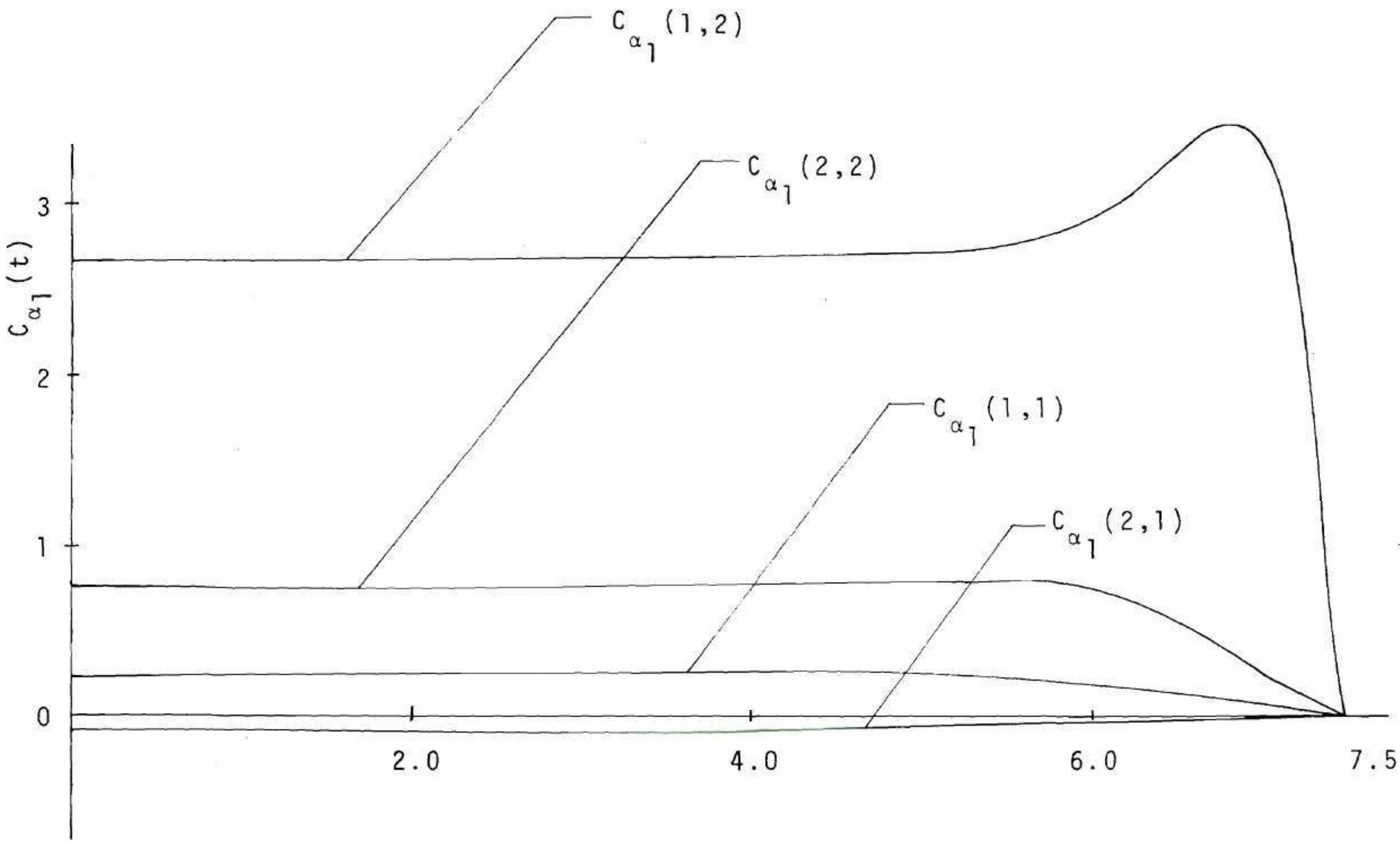

Figure 18. Sensitivity of Deterministic Controller Gain to α_1, Example 3.

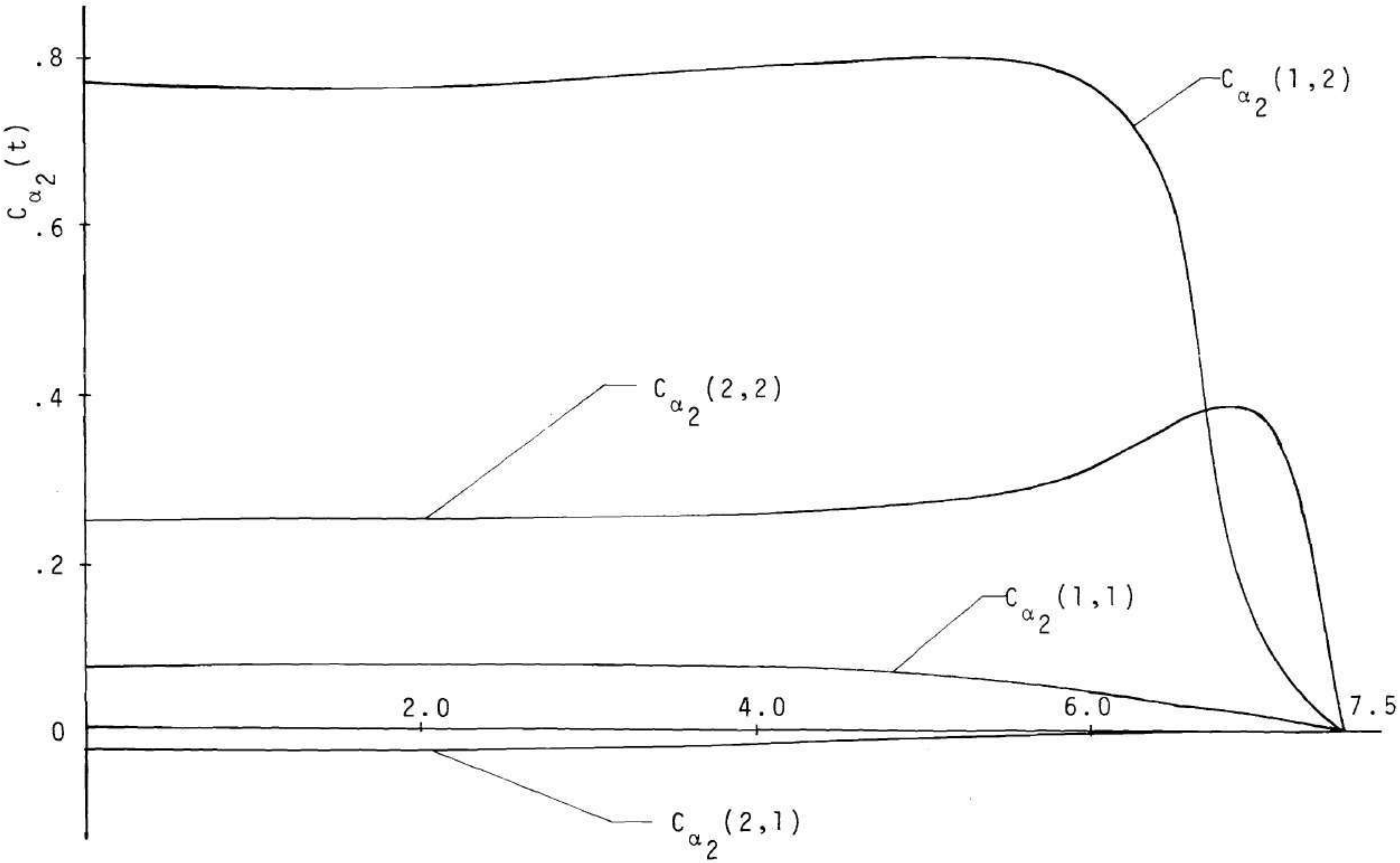

Figure 19. Sensitivity of Deterministic Controller Gain to α_2, Example 3.

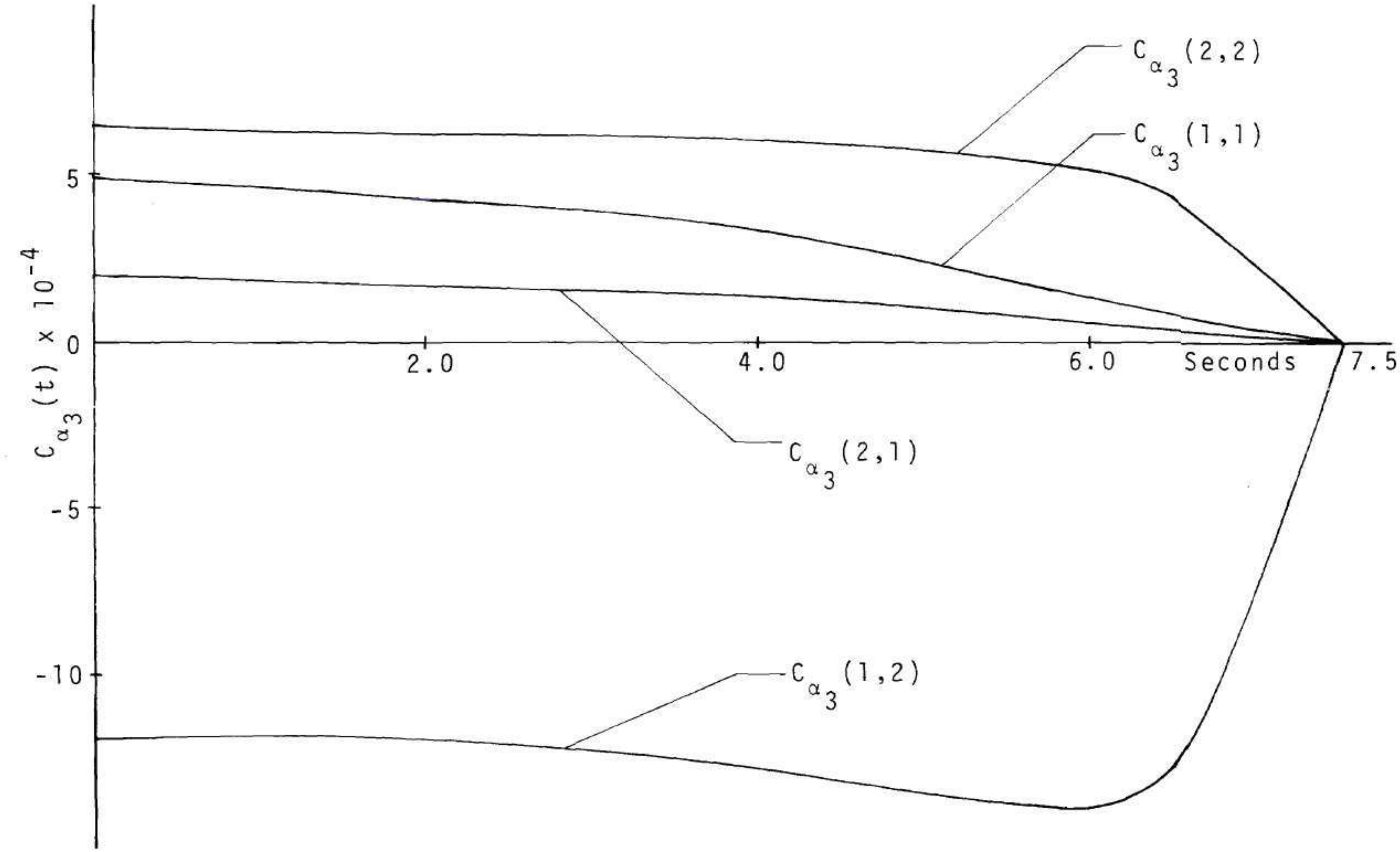

Figure 20. Sensitivity of Deterministic Controller Gain to α_3, Example 3.

$K(t)$ is obtained numerically from (4.71) and (4.72), and shown in Figure 21.

(d) Sensitivity of the Kalman Gain. Since $K(t)$ depends only on α_3, $K_{\alpha_1}(t)$ and $K_{\alpha_2}(t)$ are identically zero. Next consider the calculation of $K_{\alpha_3}(t)$. Using (2.43) and (2.44), $K_{\alpha_3}(t)$ is given by

$$K_{\alpha_3}(t) = P_{\alpha_3} H^T Q_v^{-1} , \tag{4.73}$$

$$\dot{P}_{\alpha_3} = P_{\alpha_3}[F - PH^T Q_v^{-1} H]^T + [F - PH^T Q_v^{-1} H]P_{\alpha_3} + [F_{\alpha_3} P + PF^T_{\alpha_3}],\ P_{\alpha_3}(t_o) = 0. \tag{4.74}$$

After the numerical values of F, F_{α_3}, Q_v, H and P are substituted in, (4.74) is numerically solved for $P_{\alpha_3}(t)$, and then (4.73) for K_{α_3}. $K_{\alpha_3}(t)$ is plotted in Figure 22. $K_{\alpha_3}(t)$ appears to be fairly small.

(e) MSTS of the original LSR. Using (2.54), the equation for the augmented state matrix, $P^i_{\nu\nu}(t)$, $i = 1,2,3$, is given by

$$\dot{P}^i_{\nu\nu} = \Sigma^i(t)P^i_{\nu\nu} + P^i_{\nu\nu}\Sigma^i(t) + \Omega^i,\ P^i_{\nu\nu}(t_o) = Q^i_o, \quad i=1,2,3 \tag{4.75}$$

where

$$\Sigma^i(t) = \begin{bmatrix} F & -GC & F_{\alpha_i} & -GC_{\alpha_i} - G_{\alpha_i}C \\ KH & F-GC-KH & K_{\alpha_i}H & F_{\alpha_i} - GC_{\alpha_i} - G_{\alpha_i}C \\ 0 & 0 & F & -GC \\ 0 & 0 & KH & F-GC-KH \end{bmatrix}, \tag{4.76}$$

$Q^1_o = Q^2_o = Q^3_o$ (for numerical values see page 117),

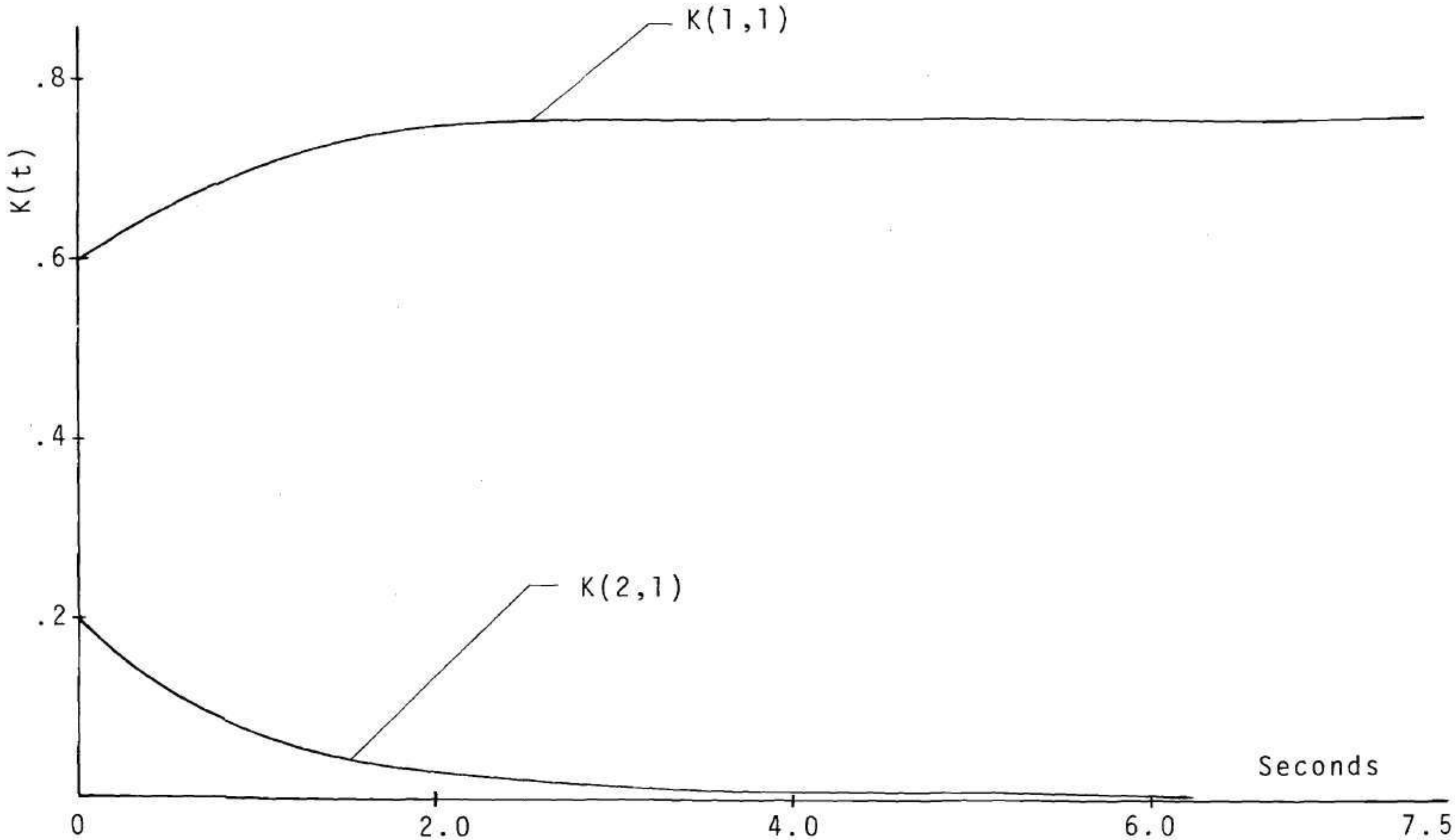

Figure 21. Kalman Gain, Example 3.

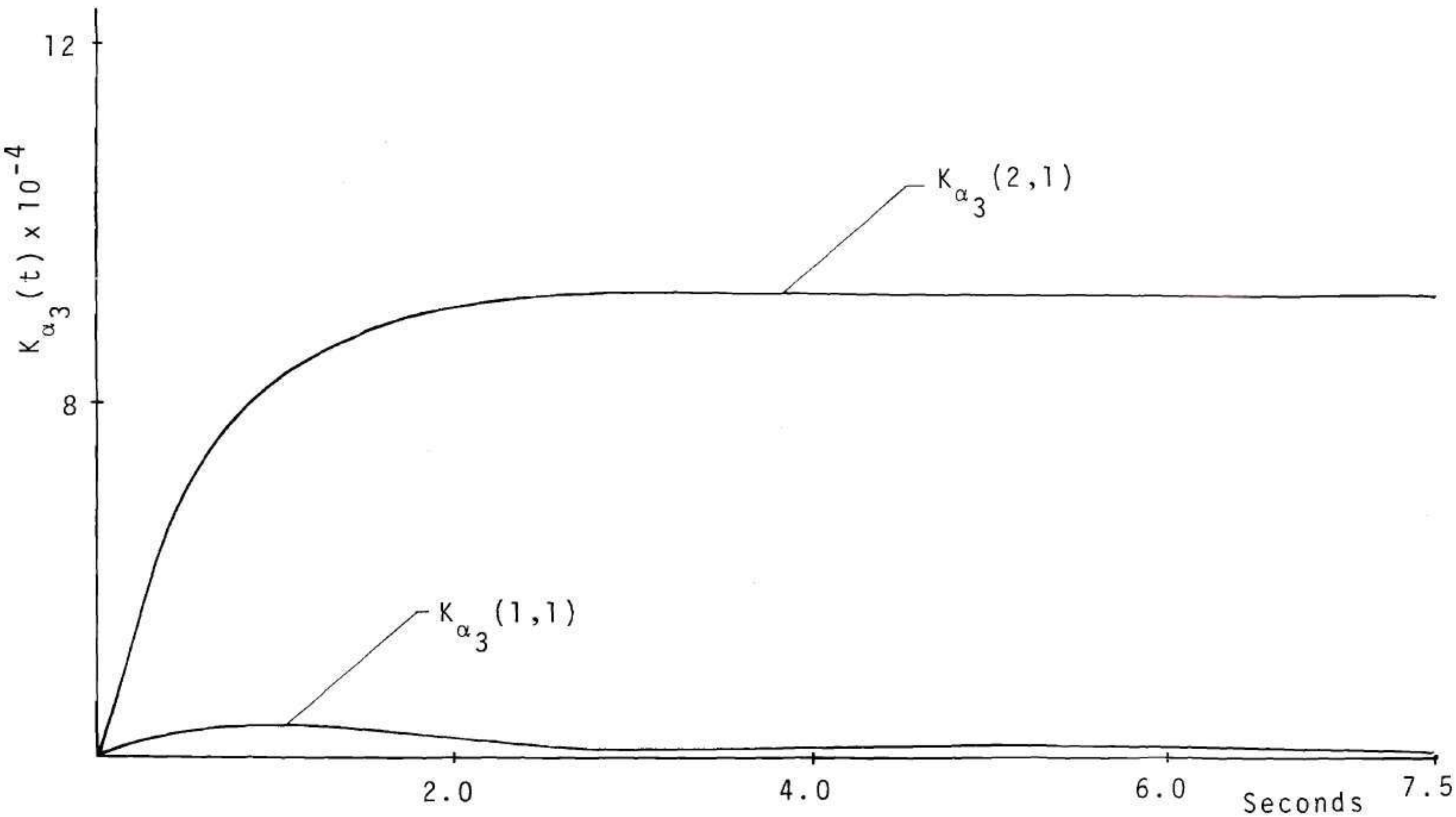

Figure 22. Sensitivity of Kalman Gain to α_3, Example 3.

$$\Omega^i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_{\alpha_i} Q_v K_{\alpha_i}^T & 0 & K_{\alpha_i} Q_v K^T \\ 0 & 0 & Q_w & 0 \\ 0 & K Q_v K_{\alpha_i}^T & 0 & K Q_v K^T \end{bmatrix} . \tag{4.77}$$

With the numerical values of F, G, H, Q_v, Q_w, F_{α_i}, C, K, C_{α_i}, and K_{α_i} (bearing in mind that $K_{\alpha_1}(t) \equiv K_{\alpha_2}(t) \equiv 0$) substituted in, (4.75) is solved for $P^i_{\nu\nu}(t)$ for i = 1,2,3. It should be noted that $P^i_{\nu\nu}(t)$ is independent of $P^j_{\nu\nu}(t)$ for $i \neq j$: i,j = 1,2,3; and that $P^i_{\zeta\zeta}$ is the (1,1)-matrix element of $P^i_{\nu\nu}$, i = 1,2,3. Based on the definitions defined earlier in (4.58) and (4.62), has

$$P^i_{\zeta_k \zeta_k} = E\{(\zeta^i_k)^2\} \tag{4.78}$$

That is, $P^i_{\zeta_k \zeta_k}$ is the mean square sensitivity of x_k to the parameter component α_i. For example, $P^2_{\zeta_1 \zeta_1}$ is the mean square sensitivity of $x_1(t)$ to α_2. Analogous notations will be used for the compensated system with the only difference being that ξ instead of ζ and θ instead of ν will be used.

$P^1_{\zeta_1 \zeta_1}$, $P^1_{\zeta_2 \zeta_2}$, $P^2_{\zeta_1 \zeta_1}$ and $P^2_{\zeta_2 \zeta_2}$ are plotted in Figures 23 - 26, respectively. $P^3_{\zeta_1 \zeta_1}$ and $P^3_{\zeta_2 \zeta_2}$ are of the order of magnitude of 10^{-8} and 10^{-7}, respectively. They are extremely small and there is no need to compensate for the modelling errors due to α_3 (or θ). Referring to Figure 23, $P^1_{\zeta_1 \zeta_1}$ has

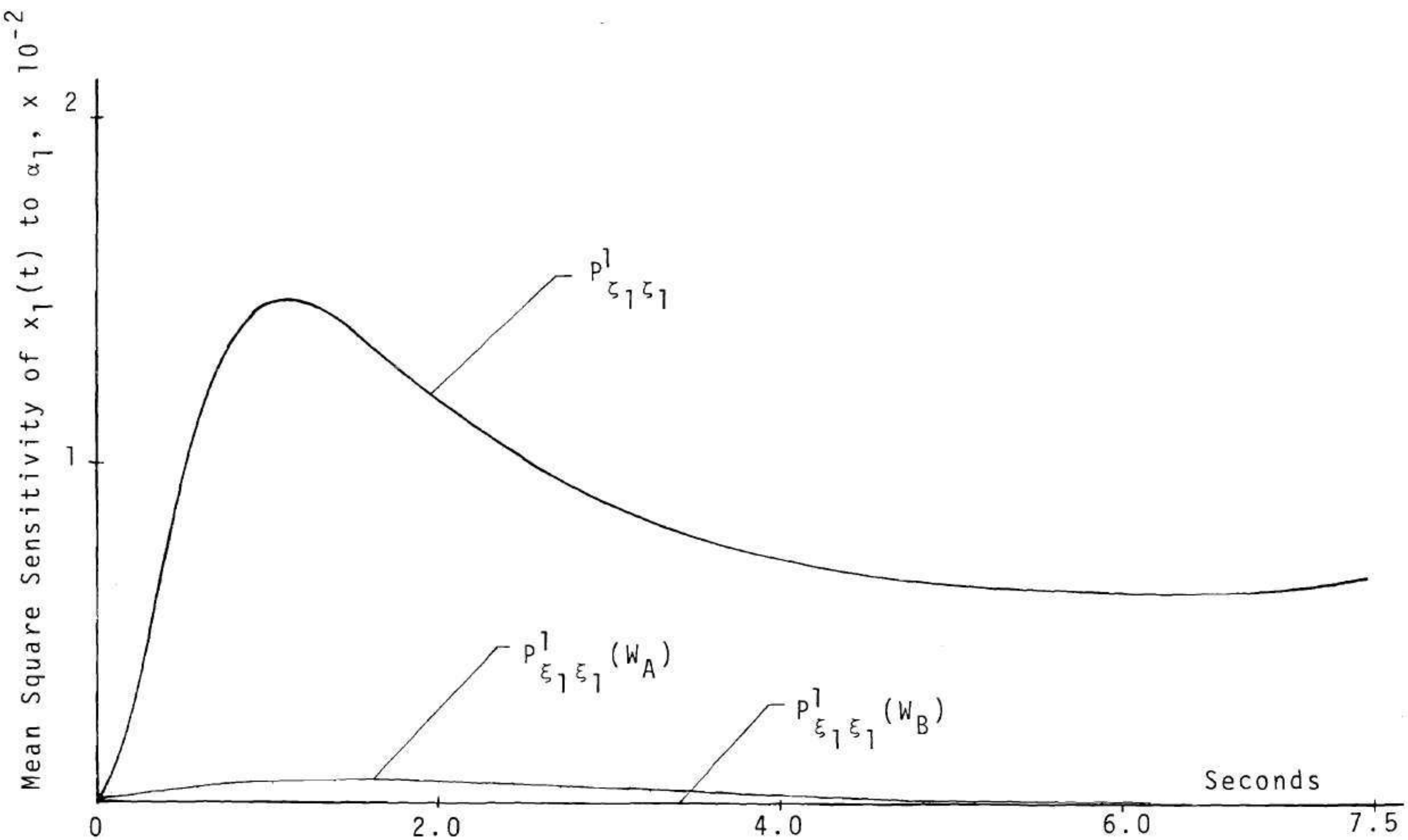

Figure 23. Mean Square Sensitivity of $x_1(t)$ of the Original LSR and the Compensated System to α_1, Example 3.

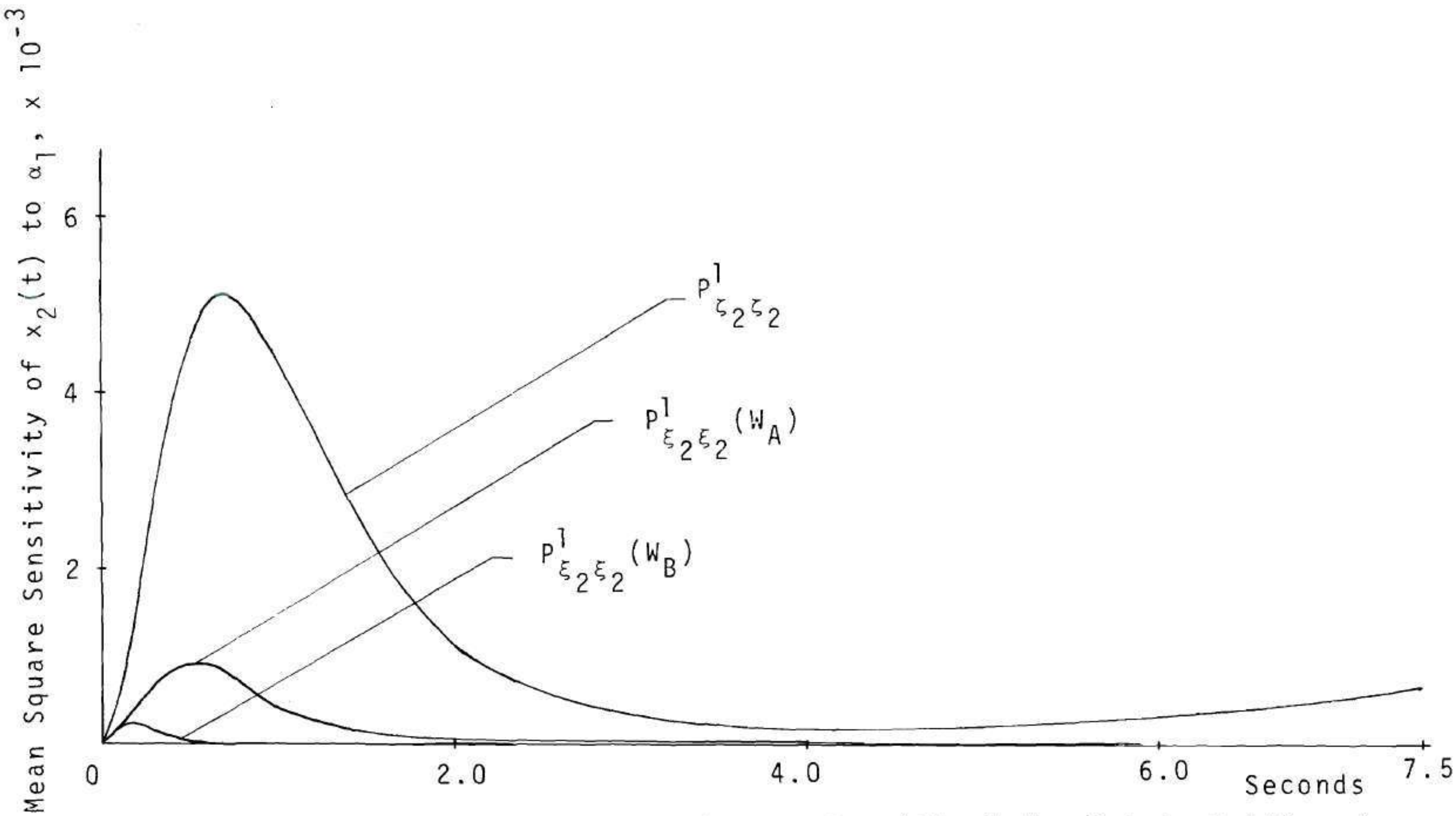

Figure 24. Mean Square Sensitivity of $x_2(t)$ of the Original LSR and the Compensated System to α_1^2, Example 3.

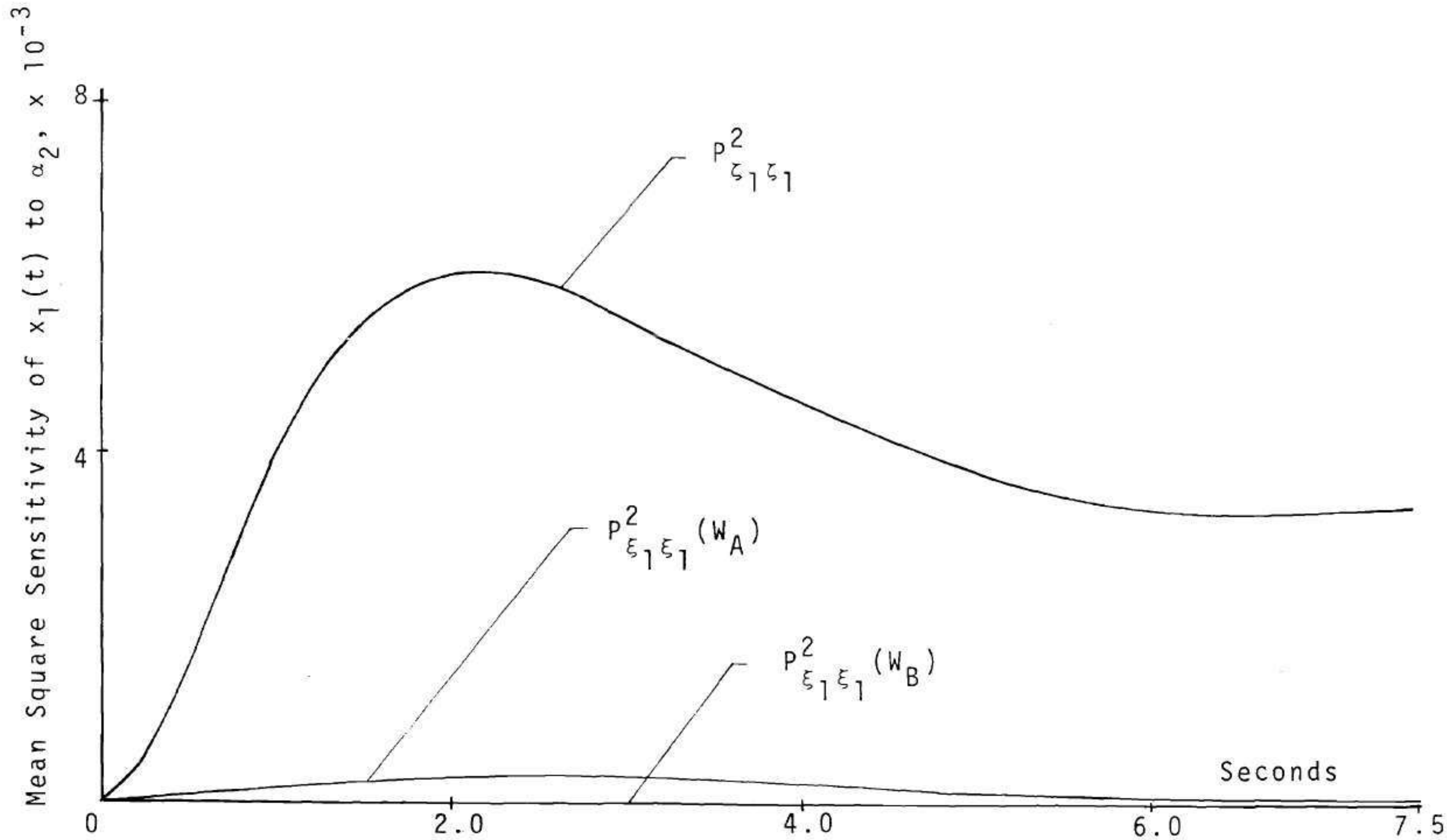

Figure 25. Mean Square Sensitivity of $x_1(t)$ of the Original LSR and the Compensated System to α_2, Example 3.

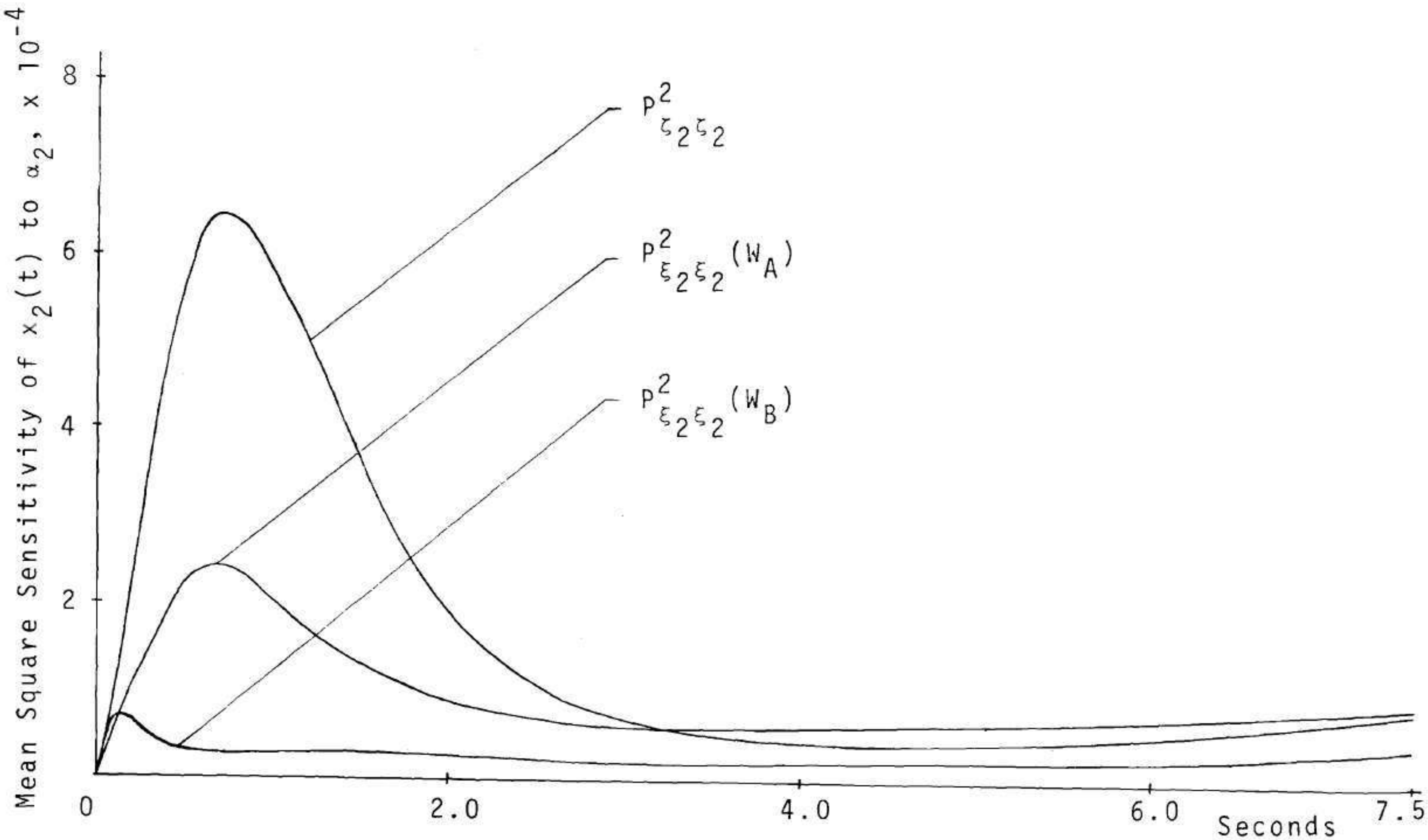

Figure 26. Mean Square Sensitivity of $x_2(t)$ of the Original LSR and the Compensated System to α_2, Example 3.

a peak value of approximately .0147 at $t = 1.2$ seconds and decreases to approximately .0068 at $t = 7.5$ seconds. $P^2_{\zeta_1\zeta_1}$ appears to be similar to $P^1_{\zeta_1\zeta_1}$ in shape but with the magnitude of about half of the latter, see Figure 25. Referring to Figures 24 and 26, $P^1_{\zeta_2\zeta_2}$ appears to be very similar in shape to $P^2_{\zeta_2\zeta_2}$ but has a peak magnitude of about 8 times of the latter. Suppose that the concentrations c_1 and c_2 vary by 30% from their nominal values. Denote the variations by Δc_1 and Δc_2, respectively. The corresponding largest approximate deviations in $\underline{x}$, denoted by $\Delta\underline{x}$, are given below.

Due to c_1: $\Delta x_1 = 0.036\ m^3$, or 7.2% or the desired steady-state volume V_0,
$\Delta x_2 = 0.043\ kmol/m^3$, or 3.4% of the desired steady-state concentration c_0.

Due to c_2: $\Delta x_1 = 0.023\ m^3$, or 4.6% of V_0,
$\Delta x_2 = 0.043\ kmol/m^3$ or 1.2% of c_0.

The figures show significant variations in $\underline{x}$ and they may be too large for some critical applications. Therefore, it is considered to be a good design practice to incorporate in the design some measure that reduces the system trajectory sensitivity to the modelling errors in c_1 and c_2. Next, it will be shown that the compensation method is highly effective in reducing the trajectory sensitivity yet at only a negligible increase in the overall cost J.

Step III: Application of the Compensation Method to Reduce the Trajectory Sensitivity.

In this step, the compensation method is applied to the LSR of this example to reduce its trajectory sensitivity to α_1 and α_2.

(a) Required Time Functions. Since C(t), K(t) and K_{α_i} (i = 1,2,3) have already been determined earlier, the functions remain to be determined are $\underline{x}_d^*(t)$ and $\underline{u}_d^*(t)$. Using (2.8) and (2.9), they are given by (4.79) and (4.80) below.

$$\dot{\underline{x}}_d^*(t) = [F - GC(t)]\underline{x}_d^*(t), \quad \underline{x}_d^*(0) = \underline{x}_o, \qquad (4.79)$$

$$\underline{u}_d^*(t) = -C(t)\underline{x}_d^*(t) . \qquad (4.80)$$

With the numerical values of F, G, C(t) and $\underline{x}_o$ from above substituted in, (4.87) and (4.80) are solved for $\underline{x}_d^*(t)$ and $\underline{u}_d^*(t)$ both of which are stored for later use.

(b) Determination of the Compensating Feedback Gain S. The auxiliary optimization problem is defined by (3.85) and (3.87) with m = 2, because only two parameters, α_1 and α_2, are to be compensated for. The necessary conditions that the gain matrix S must satisfy are equations (3.101) - (3.106). The required numerical values of the coefficient matrices and the others that have not been given before are given in the following.

$$F_{\alpha_1} = F_{\alpha_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$G_{\alpha_1} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}; \quad G_{\alpha_2} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix};$$

$$Q_f^i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1,2;$$

$$Q_o^1 = Q_o^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 10^{-4} & 0 & 0 & 0 \\ & & & & & 1.03\times10^{-2} & 0 & .01 \\ & & & & & & 0 & 0 \\ & & & & & & & .01 \end{bmatrix};$$

$$\underline{\mu}_o^1 = \underline{\mu}_o^2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad .1 \quad 0 \quad .1]^T .$$

Then, the gradient algorithm is applied to solve the TPBVP defined by (3.101) - (3.106) (see details in Chapter III) for two sets of weights defined in the following.

$$W_A = \left\{ Q_1^i,\ Q_2^i,\ i = 1,2 \,\middle|\, Q_1^1 = \begin{bmatrix} .05 & 0 \\ 0 & 1.5 \end{bmatrix},\ Q_2^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\right.$$

$$\left. Q_1^2 = \begin{bmatrix} .05 & 0 \\ 0 & 2.5 \end{bmatrix},\ Q_2^2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \right\},$$

$$W_B = \left\{ Q_1^i,\ Q_2^i,\ i = 1,2 \,\middle|\, Q_1^1 = \begin{bmatrix} 3 & 0 \\ 0 & 2.5 \end{bmatrix},\ Q_2^1 = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix}, \right.$$

$$\left. Q_1^2 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix},\ Q_2^2 = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix} \right\}.$$

W_A and W_B have to be varied by trial and error until the desired levels of trajectory sensitivity are obtained. The resulting optimal gains, $S(W_A)$ and $S(W_B)$ are shown in Figures 27 and 28, respectively. $P^1_{\xi_1\xi_1}$, $P^1_{\xi_2\xi_2}$, $P^2_{\xi_1\xi_1}$ and $P^2_{\xi_2\xi_2}$ corresponding to W_A and W_B are plotted in Figures 23 - 26, respectively. W_B penalizes the trajectory sensitivities heavier than W_A; consequently $P^1_{\xi_1\xi_1}(W_B)$, $P^1_{\xi_2\xi_2}(W_B)$, $P^2_{\xi_1\xi_1}(W_B)$, and $P^2_{\xi_2\xi_2}(W_B)$ are smaller than $P^1_{\xi_1\xi_1}(W_A)$, $P^1_{\xi_2\xi_2}(W_A)$, $P^2_{\xi_1\xi_1}(W_A)$, and $P^2_{\xi_2\xi_2}(W_A)$, respectively. An inspection of Figures 23 - 26 reveals very large trajectory sensitivity reductions afforded by the compensation method. This is achieved at only very small increase in the overall cost J for each individual result. Table 1 summarizes the results of trajectory sensitivity comparison of the original LSR with the compensated system for 30% variations of c_1 and c_2 from their nominal values.

The following notations will be used from now on.

$\Delta\underline{x}(\delta\alpha_1, 0)$ = trajectory dispersion due to $\delta\alpha_1$ alone,

$\Delta\underline{x}(0, \delta\alpha_2)$ = trajectory dispersion due to $\delta\alpha_2$ alone.

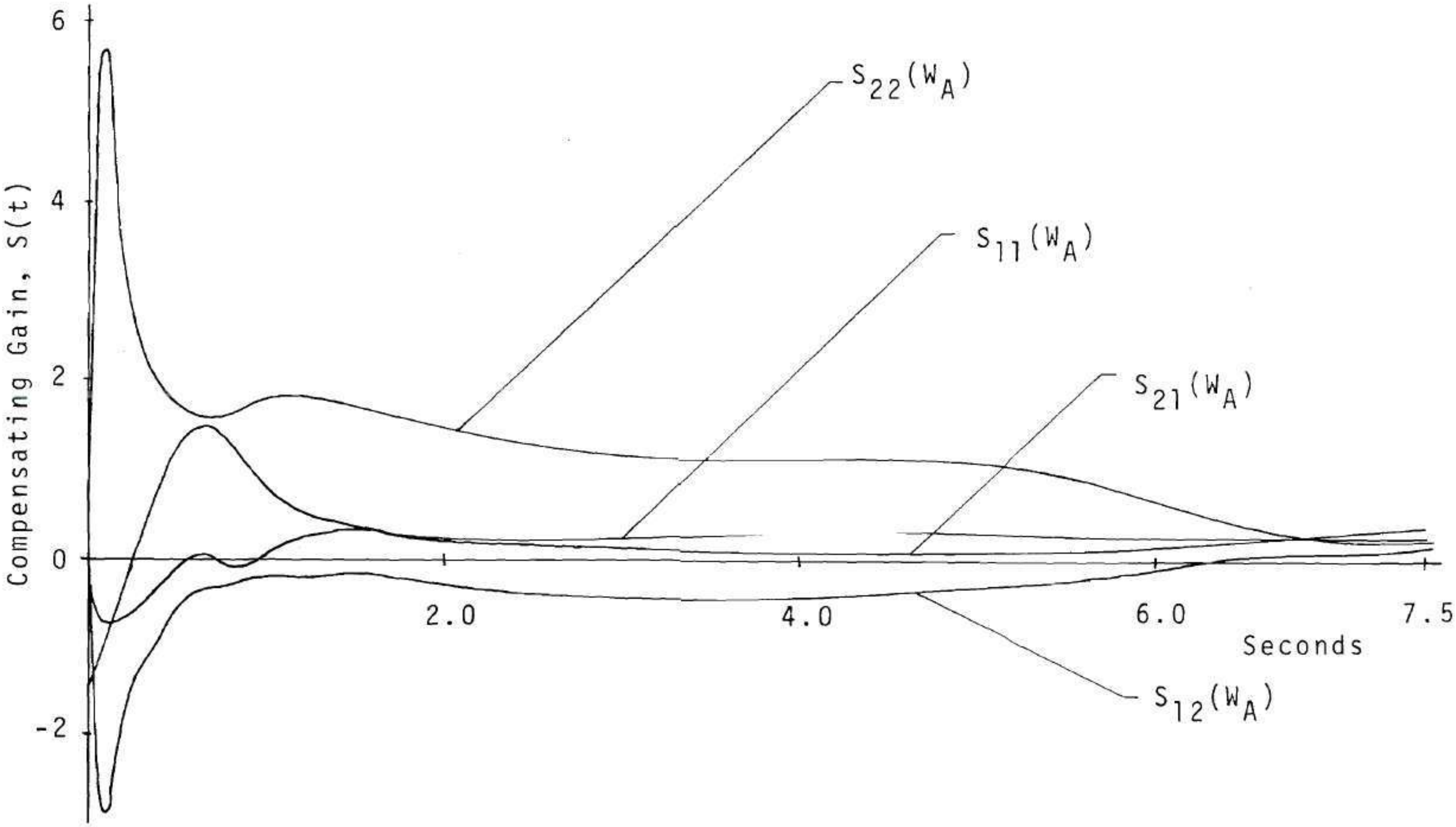

Figure 27. Compensating Gain Corresponding to Weight Set W_A, Example 3.

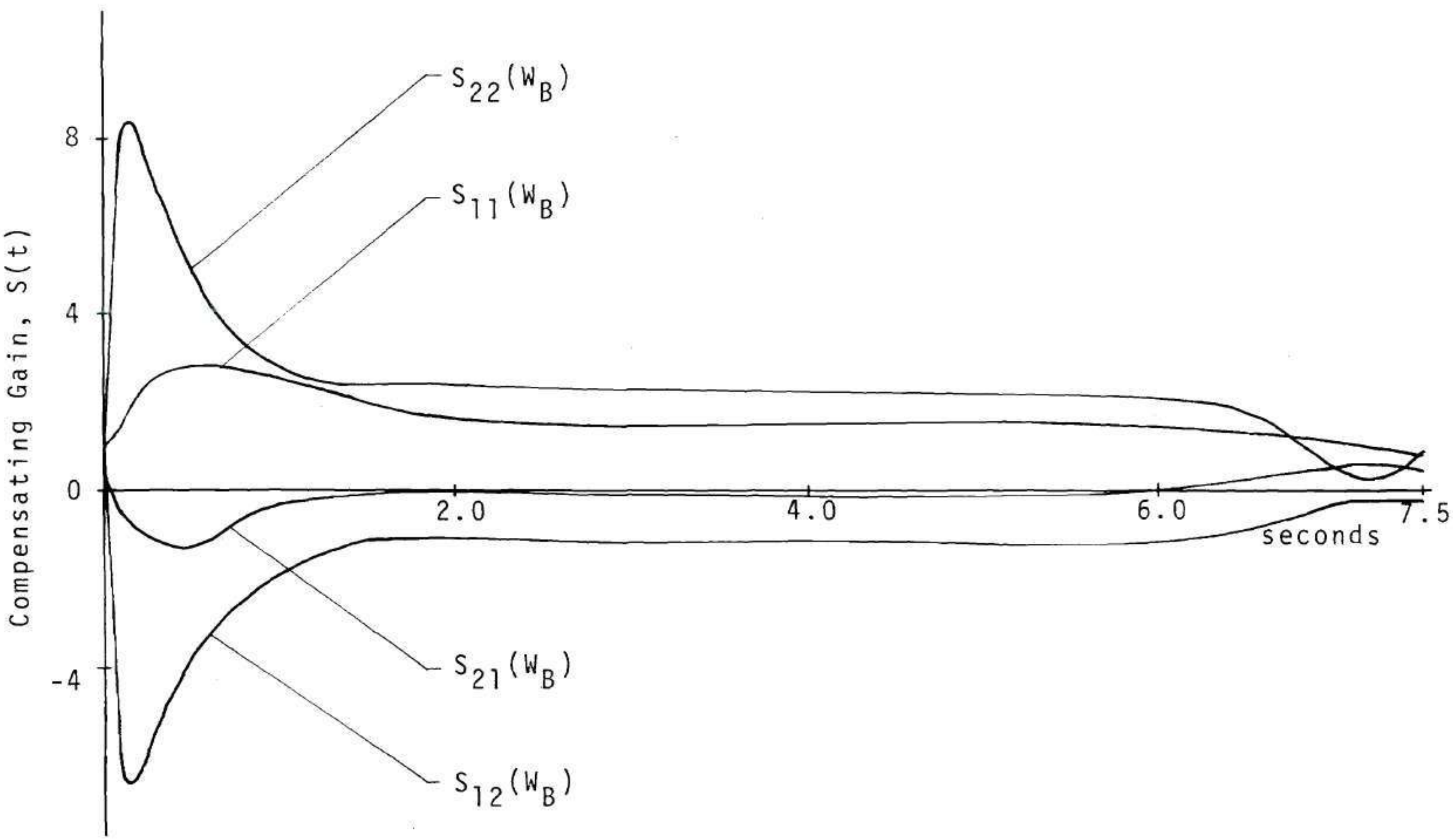

Figure 28. Compensating Gain Corresponding to Weight Set W_B, Example 3.

Table 1. Trajectory Sensitivity Comparison of the Original LSR with the Compensated System for Example 3.

Systems	30% Variation in c_1				30% Variation in c_2				Nominal Overall Cost	% Increase in overall Cost
	$\mathrm{Max}\lvert\Delta x_1(\delta\alpha_1,0)\rvert$		$\mathrm{Max}\lvert\Delta x_2(\delta\alpha_1,0)\rvert$		$\mathrm{Max}\lvert\Delta x_1(0,\delta\alpha_2)\rvert$		$\mathrm{Max}\lvert\Delta x_2(0,\delta\alpha_2)\rvert$			
	Actual	% of V_o	Actual	% of c_o	Actual	% of V_o	Actual	% of c_o		
Original LSR	.03642	7.28	.02163	1.73	.04655	9.36	.01529	1.22	1.1941810×10^{-2}	0
Compensated Syst. $S(W_A)$	.00780	1.56	.00892	0.71	.01102	2.20	.00948	0.76	1.2233533×10^{-2}	2.4
Compensated Syst. $S(W_B)$	.00166	0.33	.00481	0.38	.00162	0.32	.00500	0.40	1.2380721×10^{-2}	3.7

It is clear from Table 1 that under the influence of the modelling errors either in c_1 or in c_2, the compensated system helps to drastically reduce the trajectory dispersion, $\Delta\underline{x}(t)$. For example, the compensated system using $S(W_B)$ reduces Δx_1, caused by the modelling errors in c_1, of the original LSR from 7.28% to only 0.33% at a very small increase in the overall cost of only 3.7%. Further reductions in $\Delta\underline{x}$ are possible with larger elements of Q_1^1 and Q_1^2.

In view of the computational burden in terms of the CPU time, $S(W_A)$ and $S(W_B)$ are obtained in the 6th and 5th iterations of the gradient algorithm, respectively; each iteration takes approximately 46.62 seconds.

Example 4

In this example, a longitudinal autopilot to maintain a small vertical acceleration of an aircraft is considered [43, p. 420]. The longitudinal perturbations of an aircraft in horizontal cruising flight are reasonably well described by the second-order system (see Figure 29)

$$\dot{\underline{x}} = F\,\underline{x} + G\,u + \underline{w}, \quad E\{\underline{x}(t_0)\} = x_0 \tag{4.81}$$

where

$$\underline{x} = [x_1 \quad x_2]^T ,$$

$$F = \begin{bmatrix} -\dfrac{1}{\alpha_1} & 1 \\ -\alpha_2^2 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \alpha_2^2 \end{bmatrix},$$

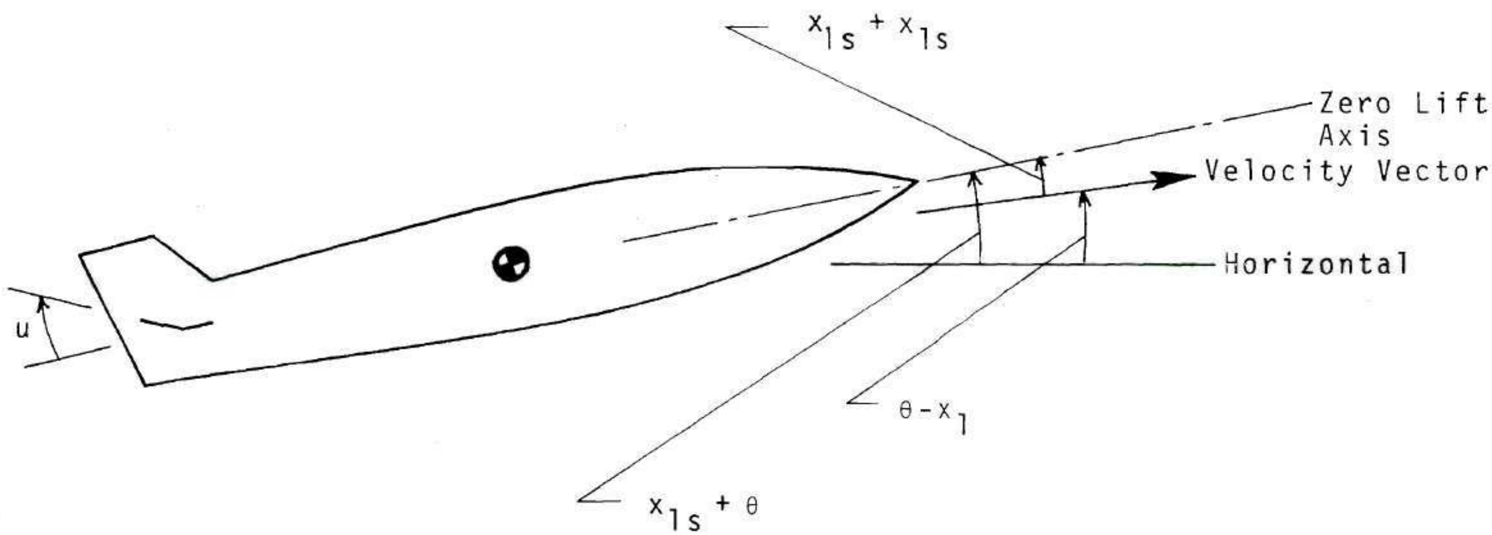

Figure 29. A Longitudinal Autopilot for Maintaining Small Vertical Acceleration of an Aircraft, Example 4.

x_1 = perturbation from cruise angle of attack, x_{1s}, in rad,

x_2 = $\dot{\theta}$, and θ is the perturbation from cruise pitch angle of the zero-lift axis, x_{1s}, in rad/sec,

α_1 = lifting time constant,

α_2 = undamped pitch natural frequency, in rad/sec.,

u = elevator deflection, in rad,

$\underline{w}$ = white Gaussian noise.

In order to achieve the design objective, the following LSR problem is considered.

$$\min_{u(t)} J = \min_{u(t)} \frac{1}{2} E\{\int_0^{1.5} [||\underline{x}(t)||_A^2 + Bu^2]dt\}, \quad (4.82)$$

subject to (4.81) with the measurement system below,

$$\underline{z} = H\underline{x} + \underline{v} . \quad (4.83)$$

Equation (4.82) is aimed at minimizing the perturbation from the cruise angle of attack, x_1, the rate of perturbation from cruise pitch angle of the zero-lift axis, $\dot{\theta}$, and the control effort. Frequently in practice, the actual values of lifting time constant α_1 and the undamped natural frequency α_2 are not known accurately. Therefore, it is expedient to perform a sensitivity analysis of the LSR and to apply the compensation method to reduce the sensitivity if it is significant. The following numerical values will be used.

Let $\underline{\alpha}_n$ denote the nominal values of the parameter vector $\underline{\alpha}$ which is defined as

$$\underline{\alpha} = [\alpha_1 \quad \alpha_2]^T. \tag{4.84}$$

$$\underline{\alpha}_n = [.1 \quad 2]^T$$

$$A = \begin{bmatrix} .05 & 0 \\ 0 & .01 \end{bmatrix}, \ B = 1, \ H = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$S_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ Q_w = \begin{bmatrix} .002 & 0 \\ 0 & .002 \end{bmatrix}, \ Q_v = .001,$$

$$\underline{x}_o = \begin{bmatrix} .5 & 0 \end{bmatrix}^T, \quad P_o = \begin{bmatrix} .0005 & 0 \\ 0 & 0 \end{bmatrix},$$

Step size of numerical integration = .01.

<u>Step I</u>: Identifications of Variables

All variables have been identified in the above.

<u>Step II</u>: Sensitivity Analysis of the Original LSR.

(a) The Deterministic Controller. The controller gain, C(t), is numerically obtained from (2.10) and (2.11) with F and G evaluated at $\underline{\alpha} = \underline{\alpha}_n$, i.e.,

$$F = \begin{bmatrix} -10 & 1 \\ -4 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

C(t) is shown in Figure 30. C(t) is stored for later use.

(b) Sensitivity of the Controller Gain. $C_{\alpha_i}(t)$, $i = 1,2$, are obtained numerically from (2.41) and (2.42) with the numerical values of F_{α_i} and G_{α_i} as given below.

$$F_{\alpha_1} = \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_{\alpha_2} = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix},$$

$$G_{\alpha_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_{\alpha_2} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

$C_{\alpha_1}(t)$ and $C_{\alpha_2}(t)$ are shown in Figure 30 together with C(t), and they are stored for later use. Let $\delta\underline{\alpha}$ be small variations of the actual values of $\underline{\alpha}$ from the nominal values, i.e.,

$$\delta\underline{\alpha} = \underline{\alpha} - \underline{\alpha}_n \tag{4.85}$$

where

$$\delta\underline{\alpha} \triangleq [\delta\alpha_1 \quad \delta\alpha_2]^T . \tag{4.86}$$

The total change in $C(\underline{\alpha},t)$ corresponding to $\delta\underline{\alpha}$ is approximately given by

$$\Delta C(t) = C_{\alpha_1}(\underline{\alpha}_n,t)\delta\alpha_1 + C_{\alpha_2}(\underline{\alpha}_n,t)\delta\alpha_2. \tag{4.87}$$

For 30% parameter variations, using (4.87) at $t = 0$ it is found that

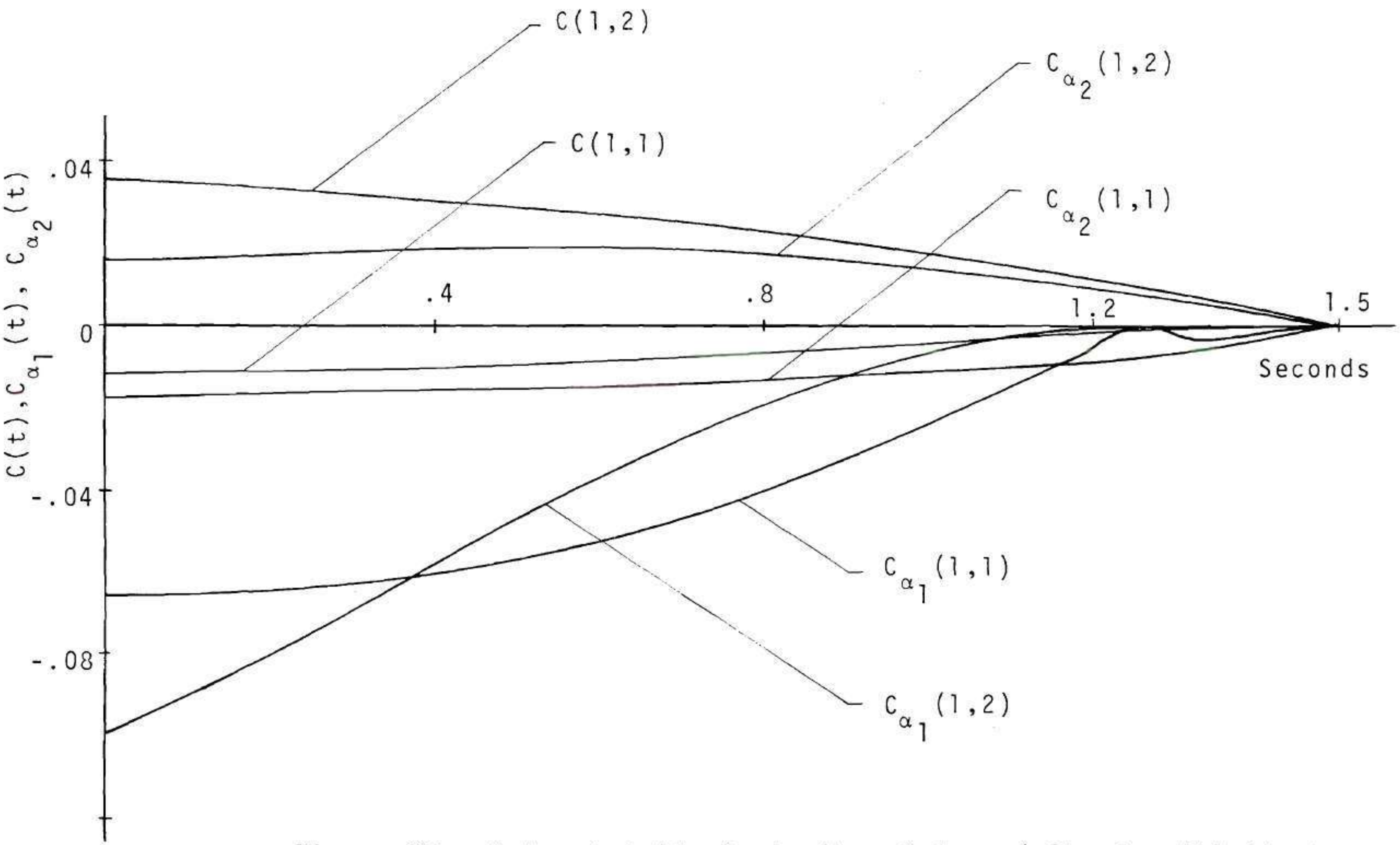

Figure 30. Deterministic Controller Gain and Its Sensitivity to α_1 and α_2, Example 4.

$$\Delta C(0) = [-.13 \quad .07]$$

which represents approximately 100% and 19% of changes in C(1,1) and C(1,2), respectively.

(c) The Kalman Filter. The Kalman filter gain, K(t), is numerically obtained from (2.28) and (2.29), and plotted in Figure 31. K(t) is also stored for later use.

(d) Sensitivity of the Kalman Gain. K_{α_i}, i = 1,2, are obtained numerically from (2.43) and (2.44). $K_{\alpha_1}(t)$ is plotted in Figure 31 together with K(t), and $K_{\alpha_2}(t)$ in Figure 32. $K_{\alpha_i}(t)$, i = 1,2, are stored for later use. $K_{\alpha_1}(t)$ is significantly larger than $K_{\alpha_2}(t)$. Using the relationship similar to (4.86), the total change in K(t), $\Delta K(t)$, at t = 0.5 is found, for the same $\delta\underline{\alpha}$ as in Step II(b), to be

$$\Delta K[0.5] = [.029 \quad .095]^T$$

which reflects 23.3% and 13.34% changes in K(1,1) and K(1,2) respectively.

(e) MSTS of the original LSR. In the following, the notations are the same as those used earlier in Example 3 $P^i_{\nu\nu}(t)$, i = 1,2 are numerically obtained from (4.75) with

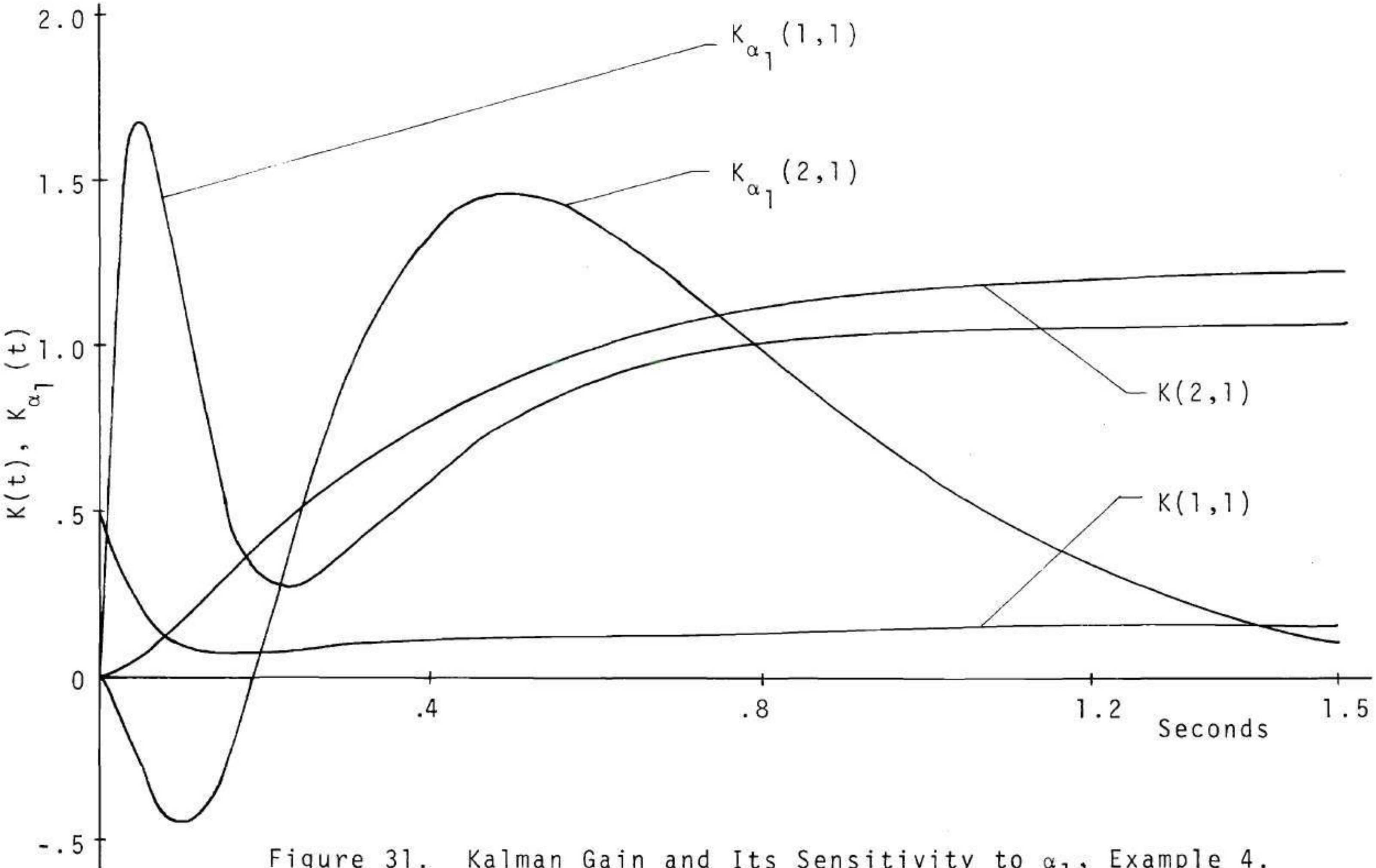

Figure 31. Kalman Gain and Its Sensitivity to α_1, Example 4.

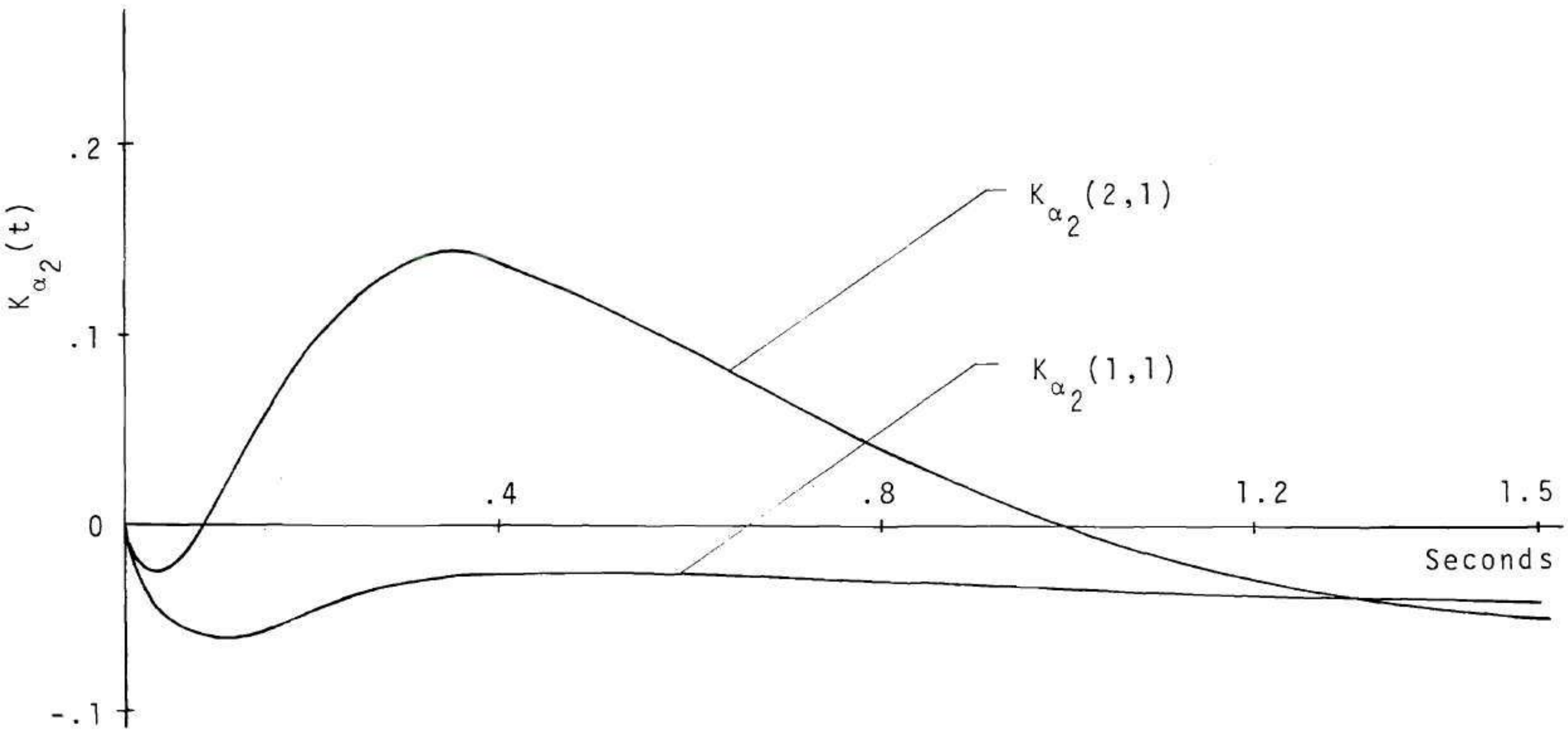

Figure 32. Sensitivity of Kalman Gain to α_2, Example 4.

$$Q_o^1 = Q_o^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & .2505 & 0 & .25 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & .25 & 0 \\ & & & & & & & 0 \end{bmatrix}.$$

$P^i_{\zeta_1\zeta_1}$, $P^i_{\zeta_2\zeta_2}$, $i = 1,2$, which are the elements of $P^i_{\nu\nu}$ are plotted in Figures 33 - 36, respectively. $P^1_{\zeta_1\zeta_1}$ has the highest peak value, i.e., approximately 3.27, which dies down very rapidly to small values within 0.4 second. The peak values of $P^1_{\zeta_2\zeta_2}$, $P^2_{\zeta_1\zeta_1}$ and $P^2_{\zeta_2\zeta_2}$ are approximately 2.42, 0.00021 and 0.024 respectively. These figures indicate that the system trajectory $\underline{x}(t)$ is more sensitive to the lifting time constant α_1 than to the undamped pitch natural frequency, bearing in mind that these figures represent mean square quantities. These figures correspond to the approximate peak magnitudes of the trajectory sensitivity of 1.81, 1.56, 0.015 and 0.16, respectively. For 30% variation in $\underline{\alpha}$ from $\underline{\alpha}_n$, the corresponding peak values of $\Delta\underline{x}$ due to $\delta\alpha_1$, alone is given by

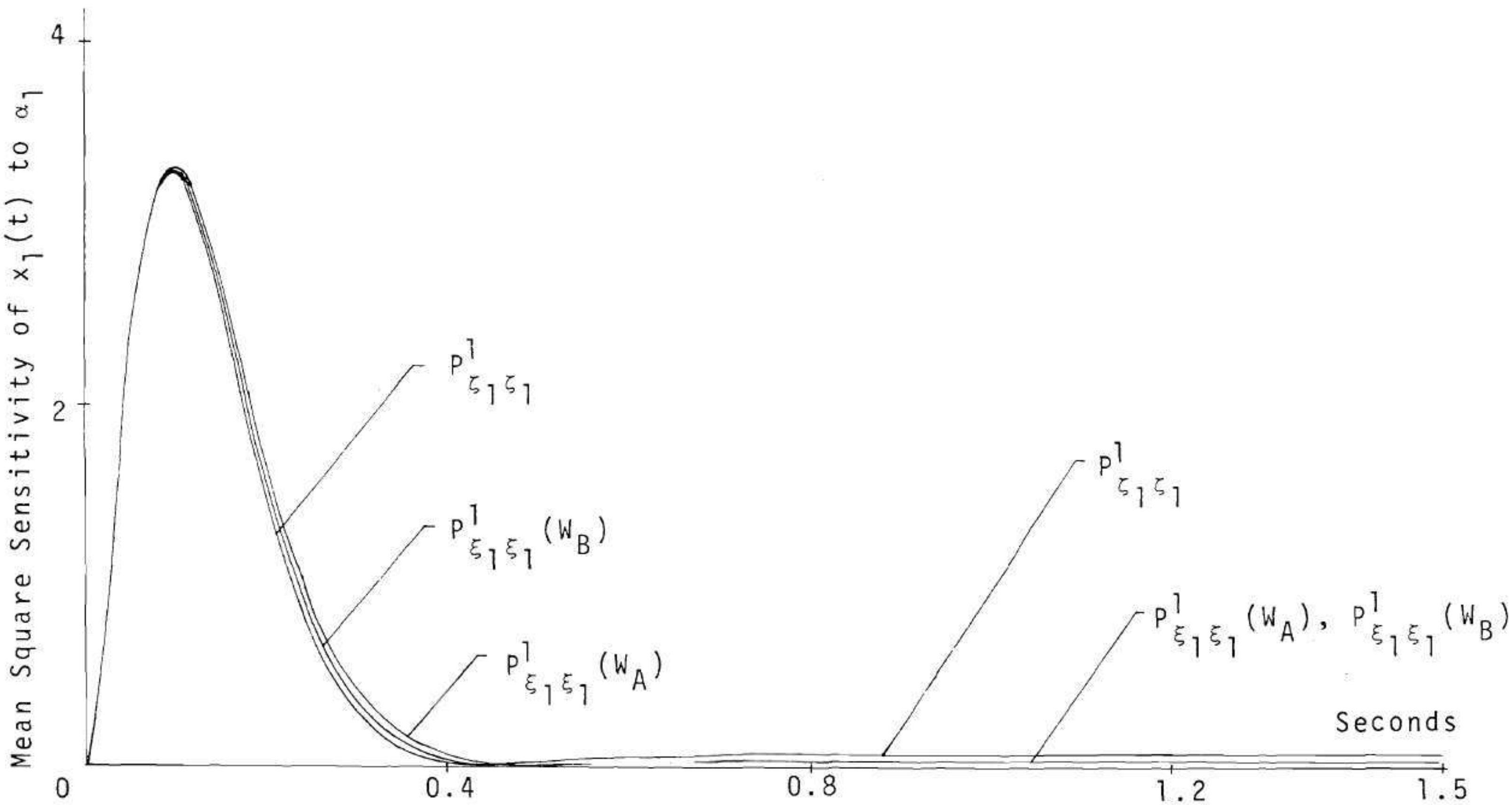

Figure 33. Mean Square Sensitivity of $x_1(t)$ of the Original LSR and the Compensated System to α_1, Example 4.

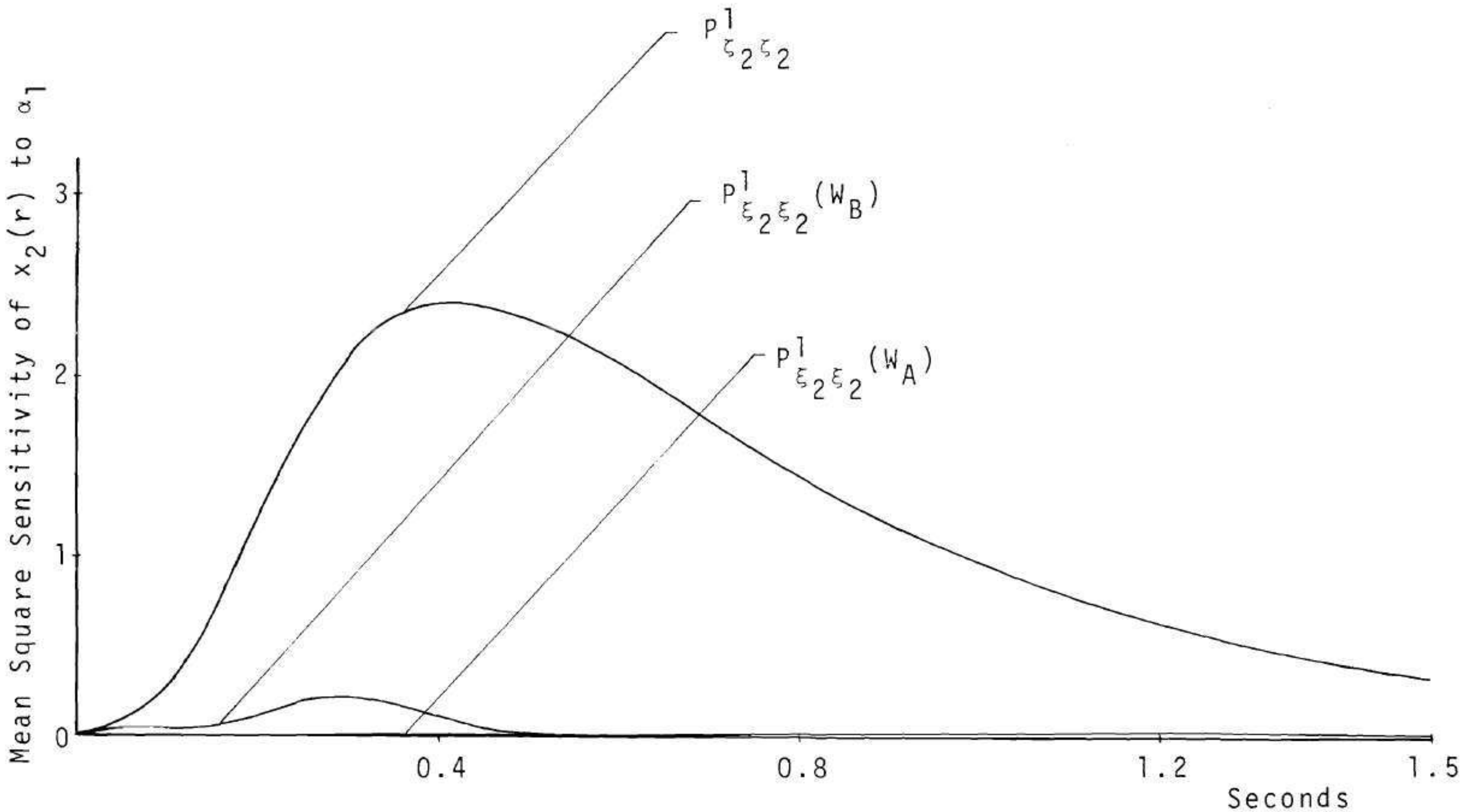

Figure 34. Mean Square Sensitivity of $x_2(2)$ of the Original LSR and the Compensated System to α_1, Example 4.

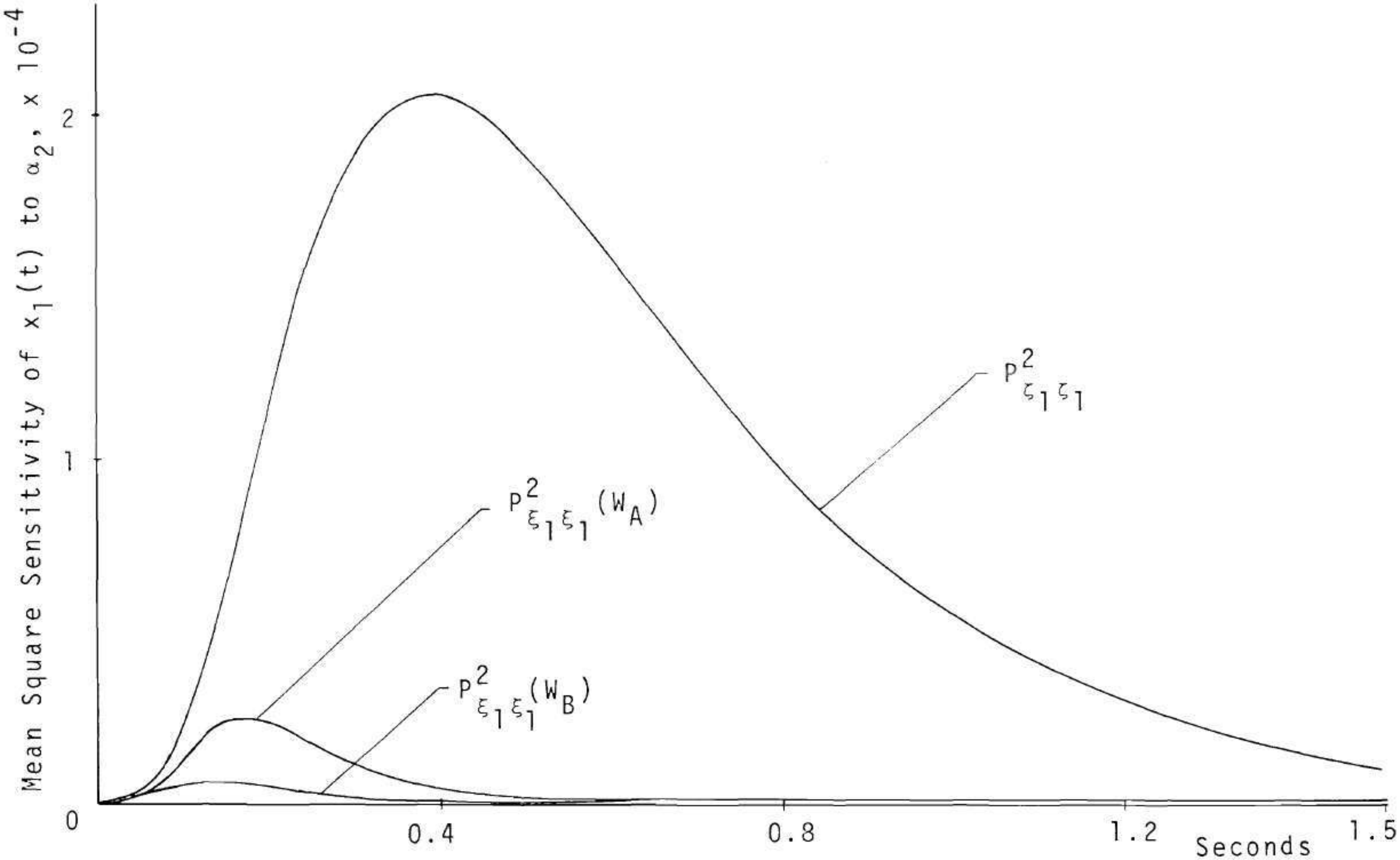

Figure 35. Mean Square Sensitivity of $x_1(t)$ of the Original LSR and the Compensated System to α_2, Example 4.

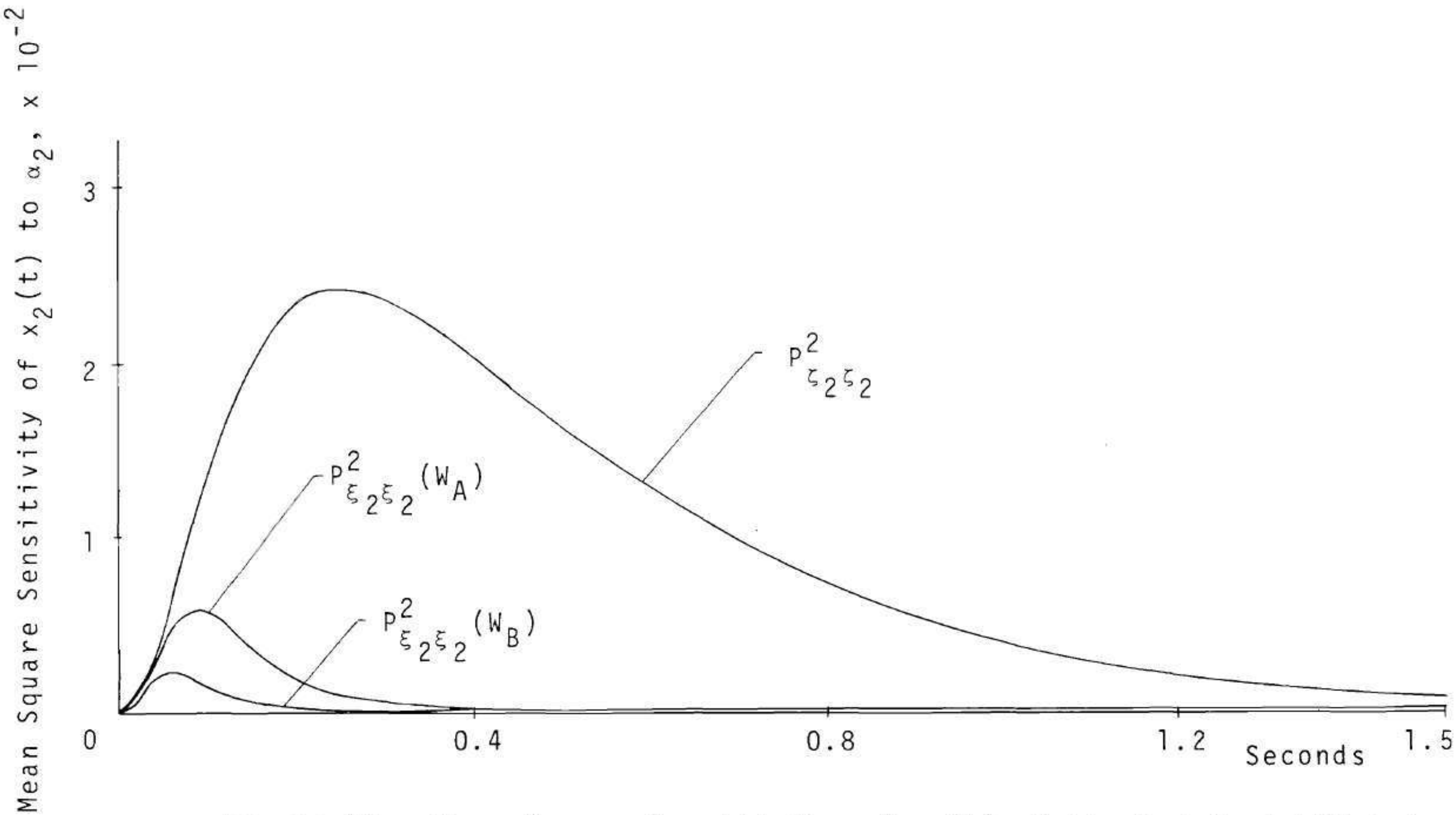

Figure 36. Mean Square Sensitivity of $x_2(t)$ of the Original LSR and the Compensated System to α_2^2, Example 4.

$$\Delta \underline{x}(\delta\alpha_1,\ 0) = [.054 \quad .047]^T , \tag{4.88}$$

and $\Delta \underline{x}$ due to $\delta\alpha_2$ alone by

$$\Delta \underline{x}(0,\ \delta x_2) = [.009 \quad .096]^T . \tag{4.89}$$

Equations (4.88) and (4.89) indicate that both $\delta\alpha_1$ and $\delta\alpha_2$ result in a fairly significant $\Delta \underline{x}$. Hence, the compensation method will be used to reduce the trajectory sensitivity due to both α_1 and α_2.

<u>Step III</u>: Application of the Compensation Method to Reduce the Trajectory Sensitivity.

(a) Required Time Functions. The time functions needed in the compensation method are $K(t)$, $K_{\alpha_1}(t)$, $K_{\alpha_2}(t)$, $\underline{x}_d^*(t)$ and $\underline{u}_d^*(t)$. The first three have been calculated in the previous steps. It remains to calculate $x_d^*(t)$ and $\underline{u}_d^*(t)$ which are given by (2.8) and (2.9).

(b) Determination of the Compensating Feedback Gain S. The TPBVP that must be solved for S is given by (3.101) - (3.106) with m = 2, because only two parameters i.e., α_1 and α_2, are considered. The TPBVP is numerically solved by the gradient algorithm (see Chapter III) for the optimal S for two different sets of weights, i.e.,

$$W_A = \left\{ Q_1^i,\ Q_2^i,\ i = 1,2 \;\middle|\; Q_1^1 = \begin{bmatrix} 10 & 0 \\ 0 & 2.5 \end{bmatrix},\ Q_2^1 = 10^{-5}, \right.$$
$$\left. Q_1^2 = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 2\text{x}10^{-4} \end{bmatrix},\ Q_2^2 = 10^{-5} \right\}, \tag{4.90}$$

$$W_B = \left\{ Q_1^i,\ Q_2^i,\ i=1,2 \;\middle|\; Q_1^1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix},\ Q_2^1 = 10^{-5} \right.$$

$$\left. Q_1^2 = \begin{bmatrix} 10^{-2} & 0 \\ 0 & 2\times 10^{-2} \end{bmatrix},\ Q_2^2 = 10^{-5} \right\} \tag{4.91}$$

As in the previous examples, W_A and W_B are obtained by trial and error. The resulting optimal gains, $S(W_A)$ and $S(W_B)$ are plotted in Figure 37. $P^1_{\xi_1\xi_1}$, $P^1_{\xi_2\xi_2}$, $P^2_{\xi_1\xi_1}$ and $P^2_{\xi_2\xi_2}$ corresponding to W_A and W_B are plotted in Figures 33-36, respectively. Considerable sensitivity reductions are achieved in all, except in $P^1_{\xi_1\xi_1}$. In Figure 33, $P^1_{\xi_1\xi_1}(W_A)$ and $P^1_{\xi_1\xi_1}(W_B)$ are slightly larger than $P^1_{\zeta_1\zeta_1}$ for $0.08 \leq t \leq 0.5$, but slightly smaller for $t > 0.5$ second. Table 2 summarizes the results of trajectory sensitivity of the original LSR and the compensated systems. The compensated systems using weight sets W_A and W_B result in increases in the overall cost of 33.5% and 98.7%, respectively. These large increases are due to unduly heavy penalties on $P^1_{\xi_1\xi_1}$ and $P^1_{\xi_2\xi_2}$. It has been observed that $P^1_{\xi_1\xi_1}$ is fairly insensitive to the penalty weight $Q_1^1(1,1)$, and that, for a given value of $Q_1^1(2,2)$ which penalizes $P^1_{\xi_2\xi_2}$, a smaller value of $Q_1^1(1,1)$ results in a smaller $P^1_{\xi_2\xi_2}$. Therefore, the overall cost of the compensated systems can be further made smaller by using smaller values of $Q_1^1(1,1)$ and $Q_1^1(2,2)$ without any significantly adverse effects on the system trajectory sensitivity.

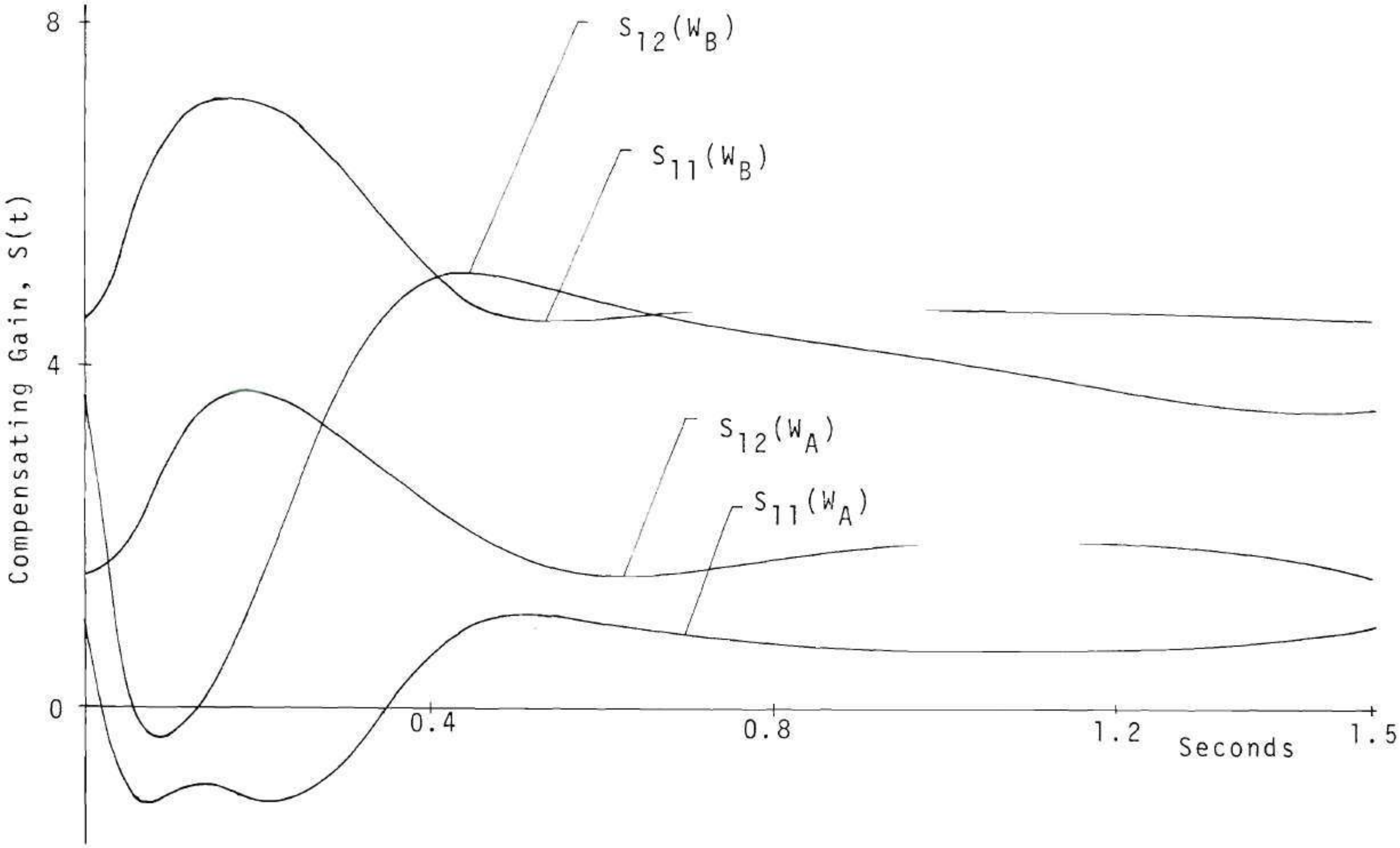

Figure 37. Compensating Gains Corresponding to Weight Sets W_A and W_B, Example 4.

Table 2. Trajectory Sensitivity Comparison of the Original LSR and the Compensated System for Example 4.

Systems	30% Variation in α_1		30% Variation in α_2		Nominal Overall Cost	% Increase in overall cost
	$\text{Max}\lvert\Delta x_1(\delta\alpha_1, 0)\rvert$	$\text{Max}\lvert\Delta x_2(\delta\alpha_1, 0)\rvert$	$\text{Max}\lvert\Delta x_1(0, \delta\alpha_2)\rvert$	$\text{Max}\lvert\Delta x_2(0, \delta\alpha_2)\rvert$		
Original LSR	.0543	.047	.009	.096	$.46774582\text{x}10^{-3}$	0
Compensated System with $S(W_A)$	.0546	.005	.003	.046	$.62564183\text{x}10^{-3}$	33.5
Compensated System with $S(W_B)$	.0544	.014	.00	.028	$.92942147\text{x}10^{-2}$	98.7

In view of the computational burden in terms of the CPU time, it takes 7 and 6 iterations of the gradient algorithm to obtain $S(W_A)$ and $S(W_B)$, respectively, and each iteration takes approximately 56 seconds.

Summary

For the majority of the examples considered, the compensation method proves to be very effective in reducing trajectory sensitivity at only a nominal increase in the overall cost. However, in Example 4 the increase in the overall cost is relatively high. This is because of the unduly heavy penalties on $P^1_{\xi_1\xi_1}$ and $P^1_{\xi_2\xi_2}$. To reduce the overall cost, the penalties should be much more relaxed. For this particular example, this can be done without any significantly adverse effects on the system trajectory sensitivity. A small increase in the overall cost afforded by the compensation method is very useful in practical applications where high overall costs are objectionable. Also, the off-line nature and the simple configuration of the compensated system make the compensation method attractive in practical applications.

CHAPTER V

COMPARISON OF COMPENSATION METHODS

In this chapter, the compensation method is compared with the other compensation methods which treat similar sensitivity problems due to small variations of deterministic constant parameters. Emphasis is placed on the compensation methods which are based on well-established practical and analytical concepts. The comparison is rather qualitative than quantitative, because a fair quantitative comparison is either too difficult or not possible, for reason to be seen later. Hence, the comparison is on the basis of the main features, computational burden in terms of the number of equations involved, advantages and disadvantages of various methods.

There are a number of methods that are used to handle the problem of modelling errors in control systems. The control systems of interest to the research are LSR's. Some methods attack the problem directly, and others indirectly. They are given in the following.

Method 1

This method relies entirely on the property of a closed-loop control system, that in many cases the closed-loop structure makes the system less sensitive to parameter errors and external disturbances. Kreindler [47] shows via a numerical example that the extent of sensitivity reduction can be controlled by adjustments of the weighting matrices in the performance index, at the expense of the desired response or frequently the cost, since usually the feedback gain has to be increased by ten or hundred or ten thousand folds to achieve some sensitivity reduction. It should be noted, however, that this desirable property is not true in general for all closed-loop systems. Kreindler indicates in [48] that in some cases the sensitivity reduction rendered by a closed-loop optimal control system is so negligible that other more direct and effective sensitivity compensation methods may have to be resorted to, if warranted by the significance of sensitivity reduction.

Method 2

This method is based on the technique of combined parameter and state estimation [52], in which an attempt is made to simultaneously estimate the actual parameter, $\underline{\alpha}$, and the state. In this technique, $\underline{\alpha}$ is adjoined to the system state to form an augmented system, and then an appropriate filtering algorithm is applied to the augmented system to simultaneously estimate $\underline{\alpha}$ and the state. This often leads to a non-linear filtering problem (e.g., when $\underline{\alpha}$ appears as a coefficient of the state in the plant dynamics) which in general is very complex, and for which no general solutions exist. Despite all these difficulties and complexities, the thus obtained estimate is still subject to estimation errors.

Method 3

This method is based on trials and errors and also on extensive Monte Carlo simulations. In a pure estimation application, modelling errors are incorporated into the process noise via a deliberate increase in the process noise covariance in order to make the filter pay more attention to the residual So far, there has been no well-established methods as to how much the increase should be. Usually the extent of the increase in the noise covariance can only be determined after extensive Monte Carlo simulations. An increase in the process noise covariance will result in an increase in the Kalman filter bandwidth. An increased bandwidth will make the filter pass more of the measurement noise; and hence, the filter will give a less accurate estimate [38]. One might suggest a more accurate modelling of the measurement noise, for example, using a coloured-noise model. However, in the majority of practical cases, the measurement noise bandwidth is much wider than the system bandwidth; consequently, an accurate model of the measurement noise offers very little help. This method appears to cure the sensitivity problem in the Kalman filter in the class of applications in which the measurements are relatively accurate (low values of the measurement noise). Since the procedure just described aims at the sensitivity reduction in the Kalman filter only, it may not be adequate to treat the sensitivity problem of an LSR, in which the deterministic controller is also sensitive to parameter errors.

Method 4

This method is due to Higginbotham [35]. Before proceeding any further, the reader is recommended to study the details of this method in Appendix C. Basically, it is an on-line method which is based on the sensitivity concept, and which compensates for the modelling errors in the deterministic controller only while leaving the modelling errors in the Kalman filter uncompensated for. This may not be adequate because the resulting compensated system feeds back the erroneous state estimate, $\hat{\underline{x}}(t)$, given by the Kalman filter, as an input to the system; any error in $\hat{\underline{x}}(t)$ would cause a degradation in the overall system characteristics, specifically the system trajectory.

Modelling errors can cause the Kalman filter to yield an estimate, $\hat{\underline{x}}(t)$, which is subject to too large an error to be of any practical use; this is known as the divergence of the Kalman filter [10]. A very interesting study by Speyer [57] reveals that modelling errors can even cause an instability of an LSR. Therefore, in view of these adverse effects of modelling errors, it is a good design practice to compensate for the modelling errors in both the Kalman filter and the deterministic controller. In addition, assumptions (h_2) and (h_3) (see Appendix C) may not be valid in general. Assumption (h_2) may be valid when the sensitivity of the Kalman $K_\alpha(t) \triangleq \left.\frac{\partial K(\underline{\alpha},t)}{\partial \underline{\alpha}}\right|_{\underline{\alpha}=\underline{\alpha}_n}$, is negligibly small so that the error in $\hat{\underline{x}}(t)$ due to modelling errors is also negligibly small. Thus, when $K_{\alpha}(t)$ is negligibly small and assumption (h_3) holds, Higginbotham's compensation method may be used to effect a trajectory sensitivity reduction, and it involves in the total number of equations of $\frac{1}{2}[(2+2m+m^2)n^2+(4+3m)n]$, where n is the plant order and m is the dimension of the parameter vector. However, this compensation method suffers two inherent disadvantages. First, since it uses a one-degree-of-freedom control, the desired trajectory characteristics and the desired sensitivity characteristics cannot be specified independently. Second, when the modelling errors are zero, the compensated system does not reduce on the average to the original system.

Method 5

This method is due to Stavroulakis and Sarachik [50]. At this point, the reader is urged to read the details of this method in Appendix C. Essentially, this method is an off-line technique in which the output feedback control is used to minimize a cost functional which provides for a trade-off between the state (which usually represents the systems error), control effort and state-estimate sensitivity. Previously the authors used the trajectory sensitivity in the cost functional and then without any mathematical justifications replaced it by the state-estimate sensitivity -- this is essentially assumption (s_2) which may not be valid in general. And, after all, the final product one wishes to minimize is the trajectory sensitivity, not the estimate sensitivity. In addition, the authors made a small error in their derivation of the compensation method in assuming that $\left.\frac{\partial \underline{n}(\underline{\alpha},t)}{\partial \underline{\alpha}}\right|_{\underline{\alpha}=\underline{\alpha}_n} = 0$, where $\underline{n}(\alpha,t)$ is the residual. As a result of this erroneous assumption, they end up with a smaller number of equations in their TPBVP. However, the correct version of their compensation method which involves a larger number of equations is presented in Appendix C. Disregarding the small error, their method is a refinement over Higginbotham's method in that the modelling errors in both the Kalman filter and the deterministic controller are being compensated for. The total number of equations involved in this method is $\frac{1}{2}(29m+4)n^2 + \frac{1}{2}(7m+2)n$. Since this method also employs a one-degree-of-freedom control, it suffers the disadvantages similar to Higginbotham's method, i.e., the desired trajectory characteristics and the desired sensitivity characteristics cannot be independently specified, and when the modelling errors are zero the compensated system does not on the average reduce to the original LSR.

Method 6

This is the method undertaken in this research. It is essentially an off-line method which is based on the sensitivity concept. It differs from Methods 4 and 5 in that it employs a two-degree-of-freedom control, chosen to directly minimize the system trajectory sensitivity which appears as a quadratic term in an integral performance index. The two-degree-of-freedom control overcomes the disadvantages of the other two sensitivity methods in that it enables a reduction in trajectory sensitivity to be effected independently of the optimal nominal trajectory, and that when the modelling errors are zero the compensated system reduces on the average to the original system. Furthermore, as compared with Methods 4 and 5, this method does not assume any unreasonable or unjustified assumptions. The number of equations involved in the TPBVP of this method is comparable to that of the TPBVP of Method 5, i.e., $(14m + 2)n^2 + (8m + 1)n$ for this method and $(14m + 2)n^2 + (3m + 1)n$ for Method 5. It also involves another $\frac{1}{2}[(2 + m)n^2 + (4 + m)n]$ equations required for $c(t)$; $K(t)$; $K_{\alpha_i}(t)$, $i = 1,2,\ldots,m$; and $\underline{x}_d^*(t)$. Hence, the total number of equations is $\frac{1}{2}[(29m + 6)n^2 + (13m + 6)n]$.

In general, all the compensation methods which are based on the sensitivity concept share a distinct advantage over the other methods in that they do not require the knowledge of the actual parameter values; this eliminates the need for an estimation or identification of the parameters, which can be very complex. Methods 4, 5 and 6 fall within the class of methods which are based on the sensitivity concept. As noted above, Method 5 is a refinement over Method 4, and Method 6 which is the compensation method undertaken in this research is superior to the other two methods. However, under certain circumstances under which assumptions (h_2) and (h_3) are valid, Method 4 is the simplest in view of the computational burden. Consider, for example, when $n = 2$ and $m = 2$, Method 4 involves in only 30 equations whereas Methods 5 and 6 involve in 140 and 168 equations, respectively; but one has to bear in mind the two main disadvantages of Methods 4 and 5 in that the desired characteristics of the trajectory sensitivity and the optimal nominal trajectory cannot be specified independently, and that when the modelling errors are zero the compensated systems do not reduce on the average to the original systems. Since, in general, Methods 5 and 6 involve in a comparable number of equations, Method 6 should always be selected because of its significant advantages over Method 5. For convenience, partial results of comparison between Methods 4, 5 and 6 are summarized in Table 1.

It has been mentioned earlier that a fair quantitative comparison between different compensation methods, particularly, Methods 4, 5, and 6, is either too difficult or not possible. This is explained in the following. In general, the trajectory sensitivity must be evaluated at a set of parameter values and along a trajectory. These are invariably the set of nominal parameter values and the nominal trajectory, because in most cases they are the only things known to a designer, for example, see (2.39), (C.35) and (3.56). (Equation (C.14) of Method 4, in which $\underline{\sigma}_i(t)$ is evaluated at the

Table 3. Partial Results of Comparisons of Different Compensation Methods

Methods	Mode of computing		Extent of compensation		*Compensated system reduces on average to original system?		Can specify sensitivity and trajectory independently?		No. of equations for m parameters
	on-line	off-line	con-troller	filter	Yes	No	Yes	No	
Higginbotham's	✓		✓			✓		✓	$\frac{1}{2}(2+2m+m^2)n^2 + \frac{1}{2}(4+3m)n$
Stavroulakis et al's		✓	✓	✓		✓		✓	$\frac{1}{2}(29m+4)n^2 + \frac{1}{2}(7m+2)n$
Yangthara's		✓	✓	✓	✓		✓		$\frac{1}{2}(29m+6)n^2+\frac{1}{2}(13m+6)n$

*Under the conditions of zero modelling errors.

nominal coefficient matrices but along the actual trajectory is not applicable because Higginbotham is dealing with the pseudo trajectory sensitivity, $\underline{\sigma}_i(t)$, not the actual trajectory sensitivity.) In other words, the trajectory sensitivity is dependent on the nominal parameters and also the nominal trajectory. Therefore, a fair comparison of trajectory sensitivities should be on the basis of the same nominal parameter values and the same nominal trajectory. In general, there is no problem in having the same nominal parameter values ; but having the same nominal trajectory in most cases is not possible, because different compensation methods usually result in different nominal trajectories even though the nominal parameters remain the same in these methods. In some compensation methods, for example, Methods 4 and 5, even within the methods themselves different weights in the performance index result in a different nominal trajectory (and also a different trajectory sensitivity).

For the sake of completeness, another method that can be used to handle the problem of modelling errors is known as the adaptive control technique. It is typically characterized by two connected elements. The first element identifies the dynamics of the process to be controlled; and the second element selects a control policy, probably with an aid of the statistical decision theory, such that the expected risk is minimized and then typically the parameters of the controller are adjusted to maintain the system performance within the tolerable limits [56]. In short, a typical adaptive system consists of three phases: identifications, stochastic optimization and modifications (frequently, a computer is needed to perform these tasks). An adaptive control system can be extremely complex and involved, it is well-suited to more difficult and more complex problems, e.g., problems with large parameter variations, random parameters, time-varying parameters, or parameters with sudden jumps in values; problems with inacccurate dynamics models other than parameter errors; and so on. In case of a problem with small parameter variations, if the models of the process to be controlled and the measurement system are sufficiently accurate except for small errors in the parameters, Method 6 can perform just as well as does an adaptive control system but with great simplifications. Of course, for the more complex problems just mentioned, an adaptive control system is the right choice, these problems are not within the scope of Method 6.

CHAPTER VI

CONCLUSIONS

This dissertation presents a new compensation method for reducing the trajectory sensitivity of a linear stochastic regulator (LSR) to a minimum level according to some integral mean square measure. It is an off-line technique which is based on the sensitivity concept, and in which a two-degree-of-freedom control structure in the form of a combined open-loop and closed-loop control is employed. The feedback path contains a compensating gain S(t) which is chosen to minimize a stochastic performance index J_s, containing quadratic terms in trajectory sensitivity functions. The main task involved in the compensation method is to obtain an optimal compensating gain S(t). Once the optimal S(t) is determined, the remaining task is simply to connect it to the other components in a simple configuration to form a compensated system. The steps involved in determining the optimal S(t) for an LSR with modelling errors in all m components of the parameter $\underline{\alpha} = [\alpha_1 \; \alpha_2 \; .....\alpha_m]^T$ are as follows. Step (1): the weight matrices for J_s, i.e., $Q^i(t)$ and $R^i(t)$, $i = 1,2,...,m$, are chosen in the manners no different from the manners in which the weight matrices A(t) and B(t) of the overall cost J of the original

LSR are chosen. Step (2): the time functions, $\underline{u}_d^*(t)$, $\underline{x}_d^*(t)$, $K(t)$, and $K_{\alpha_i}(t)$, $i = 1,2,\ldots,m$, are calculated for $t \in [t_0, t_f]$ and stored for later use. Step (3): the TPBVP is solved numerically for the optimal $S(t)$ using a steepest-descent algorithm. Step (4): if the just obtained $S(t)$ results in satisfactory reductions in the mean square trajectory sensitivity, one then proceeds to Step (5); if not, one has to repeat Steps (1) - (4). Step (5): the optimal $S(t)$ is connected, according to the block diagram in Figure 5, to form a compensated system.

The following important facts should be noted, however. Since the TPBVP represents only the necessary conditions, the existence and uniqueness of an optimal $S(t)$ are not guaranteed. However, if sufficient reduction in J_s is achieved through iterations of the gradient algorithm, the resulting gain $S(t)$ is adequate.

The control structure of the compensation method offers two distinct advantages over the existing methods. First, it enables the desired sensitivity characteristics to be effected independently of the desired trajectory characteristics; thereby a designer can meet the specifications on both the trajectory and the sensitivity simultaneously. Second, it causes the compensated system to reduce on the average to the original LSR when the modelling errors are zero. Also, another useful property of the control structure is that in many cases it results in only a negligible increase in the overall costs. The compensation method proves to be very effective in reducing the trajectory sensitivity in the four numerical examples considered.

However, the compensation method, like the other comparable methods, involves a two-point boundary value problem with a fairly large number of equations, i.e., $(14m+2)n^2 + (2m+1)n$ equations, where n and m are the plant order and the number of the parameters to be compensated for, respectively. This may limit its use to low-order systems only. To minimize the number of equations and to avoid unnecessary compensation, a sensitivity analysis of the original LSR should be performed as a preliminary design step to determine the significant trajectory sensitivity; and only the significant trajectory sensitivity should be compensated for. Also, if the procedure outlined in Chapter III is followed, in no time one has to numerically solve more than $2n(4n + 1)$ first-order differential equations simultaneously. This could extend the usefulness of the compensation method to probably fourth- or fifth-order systems. The compensation method can be further made useful to higher-order systems through the use of a suitable plant-model order reduction technique [39] - [42] to first reduce the order of the plant models, and then the compensation technique is applied to the reduced order models.

APPENDIX A

SENSITIVITY ANALYSIS OF AN LSR

In Chapter II, the sensitivity analysis of the original LSR has been given, without all the details, for the case where the modelling errors occur only in the state coefficient matrix, $F(\alpha,t)$; the control coefficient matrix, $G(\alpha,t)$, and the initial value of the error covariance matrix, $P_0(\alpha)$. However, in this Appendix additional modelling errors in the measurement matrix, $H(\alpha,t)$; the initial value of the state, $\underline{x}_0(\alpha)$; and the noise covariances $Q_w(\alpha)$ and $Q_v(\alpha)$ will be considered; and all the detailed derivations will be given.

Consider the LSR problem defined in the following.

$$\min_{u(t)} J = \min_{\underline{u}(t)} E\left\{\frac{1}{2}\| \underline{x}(t_f)\|^2_{S_f} + \frac{1}{2}\int_{t_0}^{t_f} [\| \underline{x}(t)\|^2_{A(t)} + \|\underline{u}(t)\|^2_{B(t)}]dt\right\}, \tag{A.1}$$

subject to

$$\dot{\underline{x}} = F(\alpha,t)\underline{x} + G(\alpha,t)\underline{u}(t) + \underline{w}(t);\ E\{\underline{x}(t_0)\} = \underline{x}_0(\alpha), \tag{A.2}$$

$$\underline{z} = H(\alpha,t)\underline{x} + \underline{v}(t), \tag{A.3}$$

where $\underline{x}(t)$ is a state n-vector; $\underline{u}(t)$ is a control r-vector; $\underline{z}(t)$ is a measurement ℓ-vector; S_f and $A(t)$ are symmetric positive semi-definite $n \times n$ matrices; $B(t)$ is a symmetric positive definite $r \times r$ matrix; $H(\alpha,t)$ is an $\ell \times n$ measurement matrix; t_o and t_f are specified, fixed initial and final times, respectively; $\underline{w}(t)$ and $\underline{v}(t)$ are the process and measurement white Gaussian noises, respectively, with the following properties:

$$E\{\underline{w}(t)\} = \underline{0} \ , \tag{A.4}$$

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = Q_w(\alpha,t)\delta(t-\tau), \tag{A.5}$$

$$E\{\underline{v}(t)\} = \underline{0} \ , \tag{A.6}$$

$$E\{\underline{v}(t)\underline{v}^T(\tau)\} = Q_v(\alpha,t)\delta(t-\tau), \tag{A.7}$$

$$E\{\underline{v}(t)\underline{w}^T(\tau)\} = \underline{0} \ , \tag{A.8}$$

and α is the scalar parameter. It can be shown that the optimal control is given by

$$\underline{u}(t) = -C(t)\hat{\underline{x}}(t) \ , \tag{A.9}$$

where $C(t)$, known as the deterministic controller gain, is given by

$$C(t) = B^{-1}(t)G^T(\alpha,t)S_2(t), \tag{A.10}$$

$$\dot{S}_2(t) = -S_2(t)F(\alpha,t)-F^T(\alpha,t)S_2(t)+C^T(t)B(t)C(t)-A(t),$$

$$S_2(t_f) = S_f; \tag{A.11}$$

and $\hat{\underline{x}}$ is the state estimate given by the Kalman filter whose dynamics are defined by

$$\dot{\hat{\underline{x}}} = F(\alpha,t)\hat{\underline{x}}+G(\alpha,t)\underline{u}(t)+K(t)[\underline{z}(t)-H(\alpha,t)\hat{\underline{x}}], \quad \hat{\underline{x}}(t_o)=\underline{x}_o(\alpha); \tag{A.12}$$

$$K(t) = P(t)H^T(\alpha,t)Q_v^{-1}(\alpha,t), \tag{A.13}$$

$$\dot{P}(t) = F(\alpha,t)P(t)+P(t)F^T(\alpha,t)-K(t)Q_v(\alpha,t)K^T(t)+Q_w(\alpha,t),$$
$$P(t_o) = P_o(\alpha). \tag{A.14}$$

Substituting $\underline{u}(t)$ from (A.9) in (A.2), one has

$$\dot{\underline{x}} = F(\alpha,t)\underline{x} - G(\alpha,t)C(\alpha,t)\hat{\underline{x}} + \underline{w}(t), \quad E\{\underline{x}(t_o)\} = \underline{x}_o(\alpha). \tag{A.15}$$

Substituting $\underline{u}(t)$ from (A.9) and $\underline{z}(t)$ from (A.3) in (A.12) results in

$$\dot{\hat{\underline{x}}} = K(t)H(\alpha,t)\underline{x} + [F(\alpha,t)-G(\alpha,t)C(t) -K(t)H(\alpha,t)]\hat{\underline{x}} + K(t)\underline{v}(t),$$
$$\hat{\underline{x}}(t_o) = \underline{x}_o(\alpha). \tag{A.16}$$

Define the trajectory sensitivity by

$$\underline{\zeta}(t) = \left.\frac{\partial\underline{x}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n}, \tag{A.17}$$

and the estimate sensitivity by

$$\hat{\underline{\zeta}}(t) = \left.\frac{\partial\hat{\underline{x}}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n}, \tag{A.18}$$

where α_n is the nominal value of the parameter α. The differential equation (D.E.) for $\underline{\zeta}(t)$ is obtained by partially differentiating (A.15) with respect to α and evaluate the result at $\alpha = \alpha_n$.

$$\dot{\underline{\zeta}} = F\underline{\zeta} - GC\hat{\underline{\zeta}} + F_\alpha \underline{x} - (G_\alpha C + GC_\alpha)\hat{\underline{x}}, \quad \underline{\zeta}(t_o) = \underline{\zeta}_o , \tag{A.19}$$

where the subscript α denotes a partial derivative with respect to α and evaluated at $\alpha = \alpha_n$, and all the coefficient matrices are evaluated at $\alpha = \alpha_n$. The arguments of the matrices have been and will be dropped from now on for convenience. Also, from now on the following convention will be used. All matrices without explicit arguments to indicate otherwise will be understood to be functions of time and the nominal parameter α_n. Similarly, the D.E. for $\hat{\underline{\zeta}}(t)$ can be obtained from (A.16) as

$$\begin{aligned}\dot{\hat{\underline{\zeta}}} = KH\underline{\zeta} + (F-GC-KH)\hat{\underline{\zeta}} + (K_\alpha H + KH_\alpha)\underline{x} + (F_\alpha - G_\alpha C - GC_\alpha - K_\alpha H - KH_\alpha)\hat{\underline{x}} \\ + K_\alpha \underline{v}, \quad \hat{\underline{\zeta}}(t_o) = \underline{\zeta}_o .\end{aligned} \tag{A.20}$$

In terms of the augmented state, $\underline{\nu}(t)$, which is defined by

$$\underline{\nu}(t) = [\underline{\zeta}^T \ \hat{\underline{\zeta}}^T \ \underline{x}^T \hat{\underline{x}}^T]^T, \tag{A.21}$$

equations (A.15), (A.16), (A.19) and (A.20) can be compactly expressed as

$$\dot{\underline{\nu}}(t) = \Sigma\, \underline{\nu}(t) + \underline{\gamma}(t), \quad \underline{\nu}(t_o) = \underline{\nu}_o \tag{A.22}$$

where $$\Sigma \triangleq \begin{bmatrix} F & -GC & F_\alpha & -(G_\alpha C + GC_\alpha) \\ KH & F-GC-KH & K_\alpha H + KH_\alpha & F_\alpha - G_\alpha C - GC_\alpha - K_\alpha H - KH_\alpha \\ 0 & 0 & F & -GC \\ 0 & 0 & KH & F-GC-KH \end{bmatrix}, \tag{A.23}$$

and $\underline{\gamma}(t) \triangleq [\underline{0}^T (K_\alpha \underline{v})^T \ \underline{w}^T \ (K\underline{v})^T]^T$.

The D.E.'s for the mean values of the trajectory sensitivity and the estimate sensitivity are obtained by taking the expected value of (A.22).

$$\dot{\underline{\mu}}_\nu = \Sigma \underline{\mu}_\nu, \ \underline{\mu}_\nu(t_0) = E\{\underline{\nu}_0\} \ , \tag{A.24}$$

where

$$\underline{\mu}_\nu(t) \triangleq E\{\underline{\nu}(t)\} \ .$$

Next, the D.E.'s for the mean square trajectory sensitivity, $P_{\zeta\zeta}$, and the mean square estimate sensitivity, $P_{\hat{\zeta}\hat{\zeta}}$, will be derived. Consider the augmented matrix, $P_{\nu\nu}$, defined by

$$P_{\nu\nu} = E\{\underline{\nu}(t)\underline{\nu}^T(t)\} \ . \tag{A.25}$$

Differentiating (A.25) with respect to t, one has

$$\dot{P}_{\nu\nu} = E\{\dot{\underline{\nu}}(t)\underline{\nu}^T(t) + \underline{\nu}(t)\dot{\underline{\nu}}^T(t)\} \ . \tag{A.26}$$

Substituting $\dot{\underline{\nu}}(t)$ from (A.22) in (A.26), one has the following

$$\dot{P}_{\nu\nu} = \Sigma P_{\nu\nu} + P_{\nu\nu}\Sigma^T + \Omega, \tag{A.27}$$

where $$\Omega \triangleq E\{\underline{\gamma}(t)\underline{\nu}^T(t)\} + E\{\underline{\nu}(t)\underline{\gamma}^T(t)\} \ . \tag{A.28}$$

Next, consider the evaluation of Ω. The solution of (A.22) can be expressed as

$$\underline{\nu}(t) = \Phi_\nu(t,t_0)\underline{\nu}_0 + \int_{t_0}^{t} \Phi_\nu(t,\alpha)\underline{\gamma}(\alpha)d\alpha \ , \tag{A.29}$$

where $\Phi_\nu(\cdot)$ is the transition matrix of (A.22). Hence

$$E\{\underline{\gamma}(t)\underline{\nu}^T(t)\} = E\{\underline{\gamma}(t)\underline{\nu}_0^T\Phi_\nu^T(t,t_0) + \int_{t_0}^{t}\underline{\gamma}(t)\underline{\gamma}^T(\beta)\Phi_\nu^T(t,\beta)d\beta\} \tag{A.30}$$

Consider

$$E\{\underline{\gamma}(t)\underline{\nu}_0^T\} = E\begin{bmatrix} \underline{0} \\ K_\alpha(t)\underline{v}(t) \\ \underline{w}(t) \\ K(t)\underline{v}(t) \end{bmatrix}\begin{bmatrix} \underline{\zeta}_0^T & \underline{\zeta}_0^T & \underline{x}_0^T & \underline{x}_0^T \end{bmatrix}$$

$$= E\begin{bmatrix} 0 & 0 & 0 & 0 \\ K_\alpha(t)\underline{v}(t)\underline{\zeta}_0^T & K_\alpha(t)\underline{v}(t)\zeta_0^T & K_\alpha(t)\underline{v}(t)\underline{x}_0^T & K_\alpha(t)\underline{v}(t)\underline{x}_0^T \\ \underline{w}(t)\underline{\zeta}_0^T & \underline{w}(t)\underline{\zeta}_0^T & \underline{w}(t)\underline{x}_0^T & \underline{w}(t)\underline{x}_0^T \\ K(t)\underline{v}(t)\underline{\zeta}_0^T & K(t)\underline{v}(t)\underline{\zeta}_0^T & K(t)\underline{v}(t)\underline{x}_0^T & K(t)\underline{v}(t)\underline{x}_0^T \end{bmatrix}$$

$$= 0 \ , \tag{A.31}$$

because both $\underline{w}(t)$ and $\underline{v}(t)$ are uncorrelated to the initial state $\underline{x}_0$ and $\underline{\zeta}_0$. Now, consider part of the integrand of (A.30),

$$E\{\underline{\gamma}(t)\underline{\gamma}^T(\beta)\} = E\begin{bmatrix} \underline{0} \\ K_\alpha(t)\underline{v}(t) \\ \underline{w}(t) \\ K(t)\underline{v}(t) \end{bmatrix}\begin{bmatrix} \underline{0}^T & \underline{v}^T(\beta)K_\alpha^T(\beta) & \underline{w}^T(\beta) & \underline{v}^T(\beta)K^T(\beta) \end{bmatrix} \tag{A.32}$$

Using (A.5), (A.7) and (A.8) in (A.32), one has

$$E\{\underline{\gamma}(t)\underline{\gamma}^T(\beta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_\alpha(t)Q_v(t)\delta(t-\beta)K_\alpha^T(\beta) & 0 & K_\alpha(t)Q_v(t)K^T(\beta)\delta(t-\beta) \\ 0 & 0 & Q_w(t)\delta(t-\beta) & 0 \\ 0 & K(t)Q_v(t)K_\alpha^T(\beta)\delta(t-\beta) & 0 & K(t)Q_v(t)K^T(\beta)\delta(t-\beta) \end{bmatrix} . \tag{A.33}$$

Using the results given by (A.31) and (A.33) in (A.30), one has

$$E\{\underline{\gamma}(t)\underline{\nu}^T(t)\} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}K_\alpha Q_v K_\alpha^T & 0 & \frac{1}{2} K_\alpha Q_v K^T \\ 0 & 0 & \frac{1}{2} Qw & 0 \\ 0 & \frac{1}{2} KQ_v K_\alpha^T & 0 & \frac{1}{2} KQ_v K^T \end{bmatrix} . \tag{A.34}$$

Hence, using (A.34) and the fact that

$$E\{\underline{\nu}(t)\underline{\gamma}^T(t)\} = [E\{\underline{\gamma}(t)\underline{\nu}^T(t)\}]^T \tag{A.35}$$

in (A.28), Ω is found to be

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_\alpha Q_v K_\alpha^T & 0 & K_\alpha Q_v K^T \\ 0 & 0 & Q_w & 0 \\ 0 & KQ_v K_\alpha^T & 0 & KQ_v K^T \end{bmatrix} . \tag{A.36}$$

Next, the D.E.'s will be derived for K_α and C_α. The D.E. for K_α is obtained by partially differentiating (A.13) and (A.14) with respect to α, and evaluating the result at $\alpha = \alpha_n$.

$$K_\alpha = [P_\alpha H^T + PH_\alpha^T - PH^T Q_v^{-1} \frac{\partial Q_v}{\partial \alpha}] Q_v^{-1} \quad , \tag{A.37}$$

$$\dot{P}_\alpha = [F_\alpha P + FP_\alpha] + [F_\alpha P + FP_\alpha]^T - [PH^T Q_v^{-1}(H_\alpha P + HP_\alpha)]$$
$$- [PH^T Q_v^{-1}(H_\alpha P + HP_\alpha)]^T + PH^T Q_v^{-1} \frac{\partial Q_v}{\partial \alpha} Q_v^{-1} HP + \frac{\partial Q_w}{\partial \alpha} \ , \ P_\alpha(t_o) = \frac{\partial P_o(\alpha)}{\partial \alpha} . \tag{A.38}$$

Similarly, by partially differentiating (A.10) and (A.11) with respect to α and evaluating the result at $\alpha = \alpha_n$, one has the D.E. for $C_\alpha(t)$ below.

$$C_\alpha = B^{-1}[G_\alpha^T S_2 + G^T S_{2\alpha}], \tag{A.39}$$

$$\dot{S}_{2\alpha} = -[S_2 F_\alpha + F_\alpha^T S_2] + S_2[G_\alpha B^{-1} G^T + GB^{-1} G_\alpha^T] S_2$$
$$- S_{2\alpha}[F - GB^{-1}G^T S_2] - [F - GB^{-1}G^T S_2]^T S_{2\alpha}, \ S_{2\alpha}(t_f) = 0. \tag{A.40}$$

In summary, the trajectory sensitivity of the original LSR is given by (A.22), its mean value by (A.24), and its mean square value by (A.27).

APPENDIX B

DERIVATIONS OF THE NECESSARY CONDITIONS FOR THE COMPENSATION METHOD

This Appendix presents the detailed derivations of the necessary conditions of the compensation method undertaken in this research for an LSR with general modelling errors. Modelling errors are permitted in the state coefficient matrix, $F(\alpha,t)$; the control coefficient matrix, $G(\alpha,t)$; the measurement matrix, $H(\alpha,t)$; the initial value of the error covariance matrix, $P_0(\alpha)$; the initial value of the state, $\underline{x}_0(\alpha)$; and the noise covariances, $Q_w(\alpha)$ and $Q_v(\alpha)$. The LSR problem considered is the same as that defined in Appendix A by (A.1) - (A.8).

The main idea of the compensation method is to find the control $\underline{u}(t)$ of the form,

$$\underline{u}(t) = \underline{u}_d^*(t) - S(t)[\hat{\underline{x}}(\alpha,t)-\underline{x}_d^*(t)], \tag{B.1}$$

such that the sensitivity performance index

$$J_s = E\{\|\underline{\xi}(t_f)\|^2_{Q_f} + \int_{t_o}^{t_f}[\|\underline{\xi}(t)\|^2_{Q_1} + \|S(t)\underline{\xi}(t)\|^2_{Q_2}]dt\} \tag{B.2}$$

is minimized subject to

$$\dot{\underline{\xi}} = F\underline{\xi} - GS\hat{\underline{\xi}} + F_\alpha\underline{x} - G_\alpha S\hat{\underline{x}} + G_\alpha(\underline{u}_d^* + S\underline{x}_d^*),\ \underline{\xi}(t_o) = \underline{\xi}_o; \tag{B.3}$$

$$\begin{aligned}\dot{\hat{\underline{\xi}}} = {} & KH\underline{\xi} + (F-KH-GS)\hat{\underline{\xi}} + (K_\alpha H+KH_\alpha)\underline{x}+(F_\alpha-K_\alpha H-KH_\alpha-G_\alpha S)\hat{\underline{x}}+K_\alpha\underline{v} \\ & + G_\alpha(\underline{u}_d^* + S\underline{x}_d^*),\ \hat{\underline{\xi}}(t_o) = \underline{\xi}_o;\end{aligned} \tag{B.4}$$

$$\dot{\underline{x}} = F\underline{x} - GS\hat{\underline{x}} + G(\underline{u}_d^* + S\underline{x}_d^*) + \underline{w}, \quad E\{\underline{x}(t_o)\} = \underline{x}_o(\alpha_n); \tag{B.5}$$

$$\dot{\hat{\underline{x}}} = KH\underline{x} + (F-KH-GS)\hat{\underline{x}} + K\underline{v} + G(\underline{u}_d^*+S\underline{x}_d^*), \hat{\underline{x}}(t_o) = \underline{x}_o(\alpha_n); \tag{B.6}$$

where

$$\underline{\xi}(t) \triangleq \left.\frac{\partial \underline{x}(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n}, \tag{B.7}$$

$$\hat{\underline{\xi}}(t) \triangleq \left.\frac{\partial \hat{\underline{x}}(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n}, \tag{B.8}$$

Q_f and Q_1 are symmetric positive semi-definite $n \times n$ matrices, Q_2 is a symmetric positive definite $n \times n$ matrix, $\underline{x}_d^*(t)$ and $\underline{u}_d^*(t)$ are the optimal nominal trajectory and control of the corresponding LDR; $\hat{\underline{x}}(\alpha,t)$ is the actual estimate; and α_n is the nominal value of the parameter α. Equation (B.5) is obtained by substituting $\underline{u}(t)$ from (B.1) in (A.2); and similarly equation (B.6) by substituting $\underline{u}(t)$ from (B.1) in (A.12). Equations (B.3) and (B.4) are obtained by partially differentiating (B.5) and (B.6) with respect to α, respectively, and evaluating the results at $\alpha = \alpha_n$. Also, all terms of (B.5) and (B.6) are evaluated at $\alpha = \alpha_n$.

Next, J_s in (B.2) can be converted to its deterministic equivalent as

$$J_s = \mathrm{tr}\{Q_f P_{\xi\xi}(t_f) + \int_{t_o}^{t_f} [Q_1(t)P_{\xi\xi}(t) + S^T(t)Q_2(t)S(t)P_{\xi\xi}(t)]\, dt\} \tag{B.9}$$

where

$$P_{\xi\xi}(t) \triangleq E\{\underline{\xi}(t)\underline{\xi}^T(t)\} \tag{B.10}$$

and tr is the trace of a square matrix. Consider an augmented state $\underline{\theta}(t)$ defined by

$$\underline{\theta}(t) = [\underline{\xi}^T(t)\ \hat{\underline{\xi}}^T(t)\ \underline{x}^T(t)\ \hat{\underline{x}}^T(t)]^T \ . \tag{B.11}$$

In the augmented space, the auxiliary optimization problem defined by (B.9) and (B.3) - (B.6) becomes

$$\min_{S(t)} J_S = \min_{S(t)} \operatorname{tr}\{MP_{\theta\theta}(t_f) + \int_{t_o}^{t_f} [Q(t)P_{\theta\theta}(t) + R(t)P_{\theta\theta}(t)]dt\} \tag{B.12}$$

subject to

$$\dot{\underline{\theta}} = D\underline{\theta} + \underline{\beta} \tag{B.13}$$

where

$$P_{\theta\theta}(t) \triangleq E\{\underline{\theta}(t)\underline{\theta}^T(t)\} = \begin{bmatrix} P_{\xi\xi} & P_{\xi\hat{\xi}} & P_{\xi x} & P_{\xi\hat{x}} \\ P^T_{\xi\hat{\xi}} & P_{\hat{\xi}\hat{\xi}} & P_{\hat{\xi}x} & P_{\hat{\xi}\hat{x}} \\ P^T_{\xi x} & P^T_{\hat{\xi}x} & P_{xx} & P_{x\hat{x}} \\ P^T_{\xi\hat{x}} & P^T_{\hat{\xi}\hat{x}} & P^T_{x\hat{x}} & P_{\hat{x}\hat{x}} \end{bmatrix}, \tag{B.14}$$

$$M \triangleq \begin{bmatrix} Q_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.15}$$

$$Q(t) \triangleq \begin{bmatrix} Q_1(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.16}$$

$$R(t) \triangleq \begin{bmatrix} S^T(t)Q_2(t)S(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.17}$$

$$D \triangleq \begin{bmatrix} F & -GS & F_\alpha & -G_\alpha S \\ KH & F-GS-KH & K_\alpha H + KH_\alpha & F_\alpha - K_\alpha H - KH_\alpha - G_\alpha S \\ 0 & 0 & F & -GS \\ 0 & 0 & KH & F-GS-KH \end{bmatrix}, \tag{B.18}$$

$$\underline{\beta} \triangleq \begin{bmatrix} M_2 \\ M_2 \\ M_1 + \underline{w} \\ M_1 + K\underline{v} \end{bmatrix}, \tag{B.19}$$

$$M_1 \triangleq G(u_d^* + S\underline{x}_d^*) \quad , \tag{B.20}$$

$$M_2 \triangleq G_\alpha(\underline{u}_d^* + S\underline{x}_d^*) \;, \tag{B.21}$$

$$P_{xy}(t) \triangleq E\{\underline{x}(t)\underline{y}^T(t)\}, \quad \text{for arbitrary } \underline{x}(t) \text{ and } \underline{y}(t) \tag{B.22}$$

and K_α is given by (A.37) and (A.38) in Appendix A.

Next, the differential equation for $P_{\theta\theta}(t)$ will be derived. Differentiating (B.14) with respect to t, one has

$$\dot{P}_{\theta\theta}(t) = E\{\dot{\underline{\theta}}(t)\underline{\theta}^T(t) + \underline{\theta}(t)\dot{\underline{\theta}}^T(t)\} \ . \tag{B.23}$$

Substituting $\dot{\underline{\theta}}(t)$ from (B.13) in (B.23), one has

$$\dot{P}_{\theta\theta} = DP_{\theta\theta} + P_{\theta\theta}D^T + \Gamma \ , \tag{B.24}$$

where
$$\Gamma \overset{\Delta}{=} E\{\underline{\beta}\,\underline{\theta}^T + \underline{\theta}\,\underline{\beta}^T\} \tag{B.25}$$

which will be evaluated next. Using (B.11), (B.22) and (B.19), one has

$$E\{\underline{\beta}\,\underline{\theta}^T\} = \begin{bmatrix} M_2\underline{\mu}_\xi^T & M_2\underline{\mu}_{\hat{\xi}}^T & M_2\underline{\mu}_x^T & M_2\underline{\mu}_{\hat{x}}^T \\ M_2\underline{\mu}_\xi^T & M_2\underline{\mu}_{\hat{\xi}}^T + \frac{1}{2}K_\alpha Q_v K_\alpha^T & M_2\underline{\mu}_x^T & M_2\underline{\mu}_{\hat{x}}^T + \frac{1}{2}K_\alpha Q_v K^T \\ M_1\underline{\mu}_\xi^T & M_1\underline{\mu}_{\hat{\xi}}^T & M_1\underline{\mu}_x^T + \frac{1}{2}Q_w & M_1\underline{\mu}_{\hat{x}}^T \\ M_1\underline{\mu}_\xi^T & \frac{1}{2}KQ_v K_\alpha^T + M_1\underline{\mu}_{\hat{\xi}}^T & M_1\underline{\mu}_x^T & \frac{1}{2}KQ_v K^T + M_1\underline{\mu}_{\hat{x}}^T \end{bmatrix} ; \tag{B.26}$$

where $\underline{\mu}_x(t) \overset{\Delta}{=} E\{\underline{x}(t)\}$ for an arbitrary $\underline{x}(t)$.

According to Properties 2.1 and 3.1-3.3, one has the following

$$\underline{\mu}_x = \underline{\mu}_{\hat{x}} = \underline{x}_d^*, \tag{B.27}$$

and

$$\underline{\mu}_\xi = \underline{\mu}_{\hat{\xi}} \ . \tag{B.28}$$

Using (B.27) and (B.28) in (B.26), one has

$$E\{\underline{\beta}\ \underline{\theta}^T\} = \begin{bmatrix} M_2\underline{\mu}_\xi^T & M_2\underline{\mu}_\xi^T & \frac{1}{2}\ K_\alpha Q_v K_\alpha^T & M_2\underline{x}_d^{*T} & M_2\underline{x}_d^{*T} \\ M_2\underline{\mu}_\xi^T & M_2\underline{\mu}_\xi^T + \frac{1}{2}K\ A_v K^T & & M_2\underline{x}_d^{*T} & M_2\underline{x}_d^{*T} + \frac{1}{2}K_\alpha Q_v K^T \\ M_1\underline{\mu}_\xi^T & M_1\underline{\mu}_\xi^T & & M_1\underline{x}_d^{*T} + \frac{1}{2}Q_w & M_1\underline{x}_d^{*T} \\ M_1\underline{\mu}_\xi^T & \frac{1}{2}KQ_vK_\alpha^T + M_1\underline{\mu}_\xi^T & & M_1\underline{x}_d^{*T} & \frac{1}{2}KQ_vK^T + M_1\underline{x}_d^{*T} \end{bmatrix}, \tag{B.29}$$

The advantage of using (B.27) and (B.28) is that in the end one has to generate only $\underline{\mu}_\xi$($\underline{x}_d^*$ has already been generated for use in (B.1)) instead of all $\underline{\mu}_x$, $\underline{\mu}_\xi$, $\underline{\mu}_{\hat{x}}$ and $\underline{\mu}_{\hat{\xi}}$; this type of saving has been previously discussed in Chapter III. Since,

$$E\{\underline{\theta}\ \underline{\beta}^T\} = [E\{\underline{\beta}\ \underline{\theta}^T\}]^T , \tag{B.30}$$

and also using (B.29), Γ from (B.25) now becomes

$$\Gamma = \begin{bmatrix} C_1 & C_1 & C_2 & C_2 \\ C_1^T & C_4 & C_2 & C_3 \\ C_2^T & C_2^T & C_6 & C_5 \\ C_2^T & C_3^T & C_5^T & C_7 \end{bmatrix} \tag{B.31}$$

where

$$C_1 \triangleq M_2\underline{\mu}_\xi^T + \underline{\mu}_\xi M_2^T ,$$

$$C_2 \triangleq M_2\underline{x}_d^{*T} + \underline{\mu}_\xi M_1^T ,$$

$$C_3 \triangleq C_2 + K_\alpha Q_v K^T ,$$

$$C_4 \triangleq C_1 + K_\alpha Q_v K_\alpha^T ,$$

$$C_5 \triangleq M_1 \underline{x}_d^{*T} + \underline{x}_d^* M_1^T$$

$$C_6 \triangleq C_5 + Q_w ,$$

$$C_7 \triangleq C_5 + KQ_vK^T ,$$

$\underline{\mu}_x$ and $\underline{\mu}_\xi$ are the solutions of

$$\dot{\underline{\mu}} = D\underline{\mu}_\theta + \overline{\underline{\beta}} , \tag{B.32}$$

which results after taking the expectation of (B.13),

where $$\overline{\underline{\beta}} \triangleq [M_2^T \quad M_2^T \quad M_1^T \quad M_1^T]^T . \tag{B.33}$$

It can be shown that Γ can also be expressed as

$$\Gamma = \overline{\underline{\beta}}\,\underline{\mu}_\theta^T + \underline{\mu}_\theta \overline{\underline{\beta}}^T + \Omega \tag{B.34}$$

where Ω is given by (A.36).

Hence, the final form of the auxiliary optimization problem is as follows.

$$\min_{S(t)} J_s = \min_{S(t)} \{\mathrm{tr}\ MP_{\theta\theta}(t_f) + \int_{t_0}^{t_f} [Q(t)P_{\theta\theta} + R(t)P_{\theta\theta}(t)]dt\} \tag{B.35}$$

subject to (B.24) and (B.34). In the following, the Matrix Minimum Principle [45] will be used to solve this optimization problem. Define the Hamiltonian, $\hat{H}$, by

$$\hat{H} = \mathrm{tr}\,[(Q + R(S))P_{\theta\theta} + D(S)P_{\theta\theta}\Lambda^T + P_{\theta\theta}D^T(S)\Lambda^T$$
$$+ \Gamma(S)\Gamma^T + \underline{\lambda}\ \underline{\mu}_\theta^T D^T + \underline{\lambda}\ \underline{\overline{\beta}}^T] \tag{B.36}$$

where $\underline{\lambda}$ is a 4n costate vector corresponding to $\underline{\mu}_\theta$ with

$$\underline{\lambda} = [\underline{\lambda}_\xi^T \quad \underline{\lambda}_{\hat{\xi}}^T \quad \underline{\lambda}_x^T \quad \underline{\lambda}_{\hat{x}}^T]^T \tag{B.37}$$

and Λ is a $4n \times 4n$ symmetric costate matrix corresponding to $P_{\theta\theta}$ with

$$\Lambda = \begin{bmatrix} \Lambda_{\xi\xi} & \Lambda_{\xi\hat{\xi}} & \Lambda_{\xi x} & \Lambda_{\xi\hat{x}} \\ \Lambda_{\xi\hat{\xi}}^T & \Lambda_{\hat{\xi}\hat{\xi}} & \Lambda_{\hat{\xi}x} & \Lambda_{\hat{\xi}\hat{x}} \\ \Lambda_{\xi x}^T & \Lambda_{\hat{\xi}x}^T & \Lambda_{xx} & \Lambda_{x\hat{x}} \\ \Lambda_{\xi\hat{x}}^T & \Lambda_{\hat{\xi}\hat{x}}^T & \Lambda_{x\hat{x}}^T & \Lambda_{\hat{x}\hat{x}} \end{bmatrix}. \tag{B.38}$$

The necessary conditions given by the Matrix Minimum Principle are

(i) $$\frac{\partial H}{\partial P_{\theta\theta}} = -\dot{\Lambda}\ , \quad \frac{\partial H}{\partial \underline{\mu}_\theta} = -\dot{\underline{\lambda}}\ ; \tag{B.39}$$

(ii) $$\frac{\partial H}{\partial \Lambda} = \dot{P}_{\theta\theta}\ , \quad \frac{\partial H}{\partial \lambda} = \dot{\underline{\mu}}_\theta\ ; \tag{B.40}$$

(iii) $$\frac{\partial H}{\partial S} = 0\ , \tag{B.41}$$

and (iv) the transversality condition

$$\Lambda(t_f) = \frac{\partial}{\partial P_{\theta\theta}(t_f)}\ \mathrm{tr}[MP_{\theta\theta}(t_f)] = M^T, \tag{B.42}$$

$$\dot{\underline{\lambda}}(t_f) = \underline{0}\ . \tag{B.43}$$

Using (B.36) in (B.39) results in the costate equations

$$\dot{\Lambda} = -D^T\Lambda - \Lambda D - Q - R \ , \qquad (B.44)$$

$$\dot{\underline{\lambda}} = -D^T\underline{\lambda} - 2\Lambda\underline{\overline{\beta}} \qquad (B.45)$$

Using (B.36) in (B.40) results in the state equations

$$\dot{P}_{\theta\theta} = DP_{\theta\theta} + P_{\theta\theta}D^T + \Gamma, \ P_{\theta\theta}(t_o) = Q_o; \qquad (B.46)$$

$$\dot{\underline{\mu}}_\theta = D\underline{\mu}_\theta + \underline{\overline{\beta}} \ , \ \ \underline{\mu}_\theta(t_o) = \underline{\mu}_o \ , \qquad (B.47)$$

where Q_o and $\underline{\mu}_o$ are known initial values of $P_{\theta\theta}$ and $\underline{\mu}_\theta$, respectively. Using (B.36) in (B.40), one has

$$\frac{\partial}{\partial S} tr[(Q + R(S))P_{\theta\theta} + D(S)P_{\theta\theta}\Lambda^T + P_{\theta\theta}D^T(S)\Lambda^T + \Gamma(S)\Lambda^T + \underline{\lambda}\ \underline{\mu}_\theta^T D^T + \underline{\lambda}\ \underline{\overline{\beta}}^T] = 0. \qquad (B.48)$$

Next, the L.H.S. of (B.48) will be evaluated. First, consider the first term in the L.H.S. of (B.48).

$$\frac{\partial}{\partial S} tr[(Q+R(S))P_{\theta\theta}] = \frac{\partial}{\partial S} tr[R(S)P_{\theta\theta}] \ ,$$

$$= \frac{\partial}{\partial S} tr[S^T Q_2 S P_{\xi\xi}], \text{ using (B.14) and (B.17)},$$

$$= 2Q_2SP_{\xi\xi} \ . \qquad (B.49)$$

Next, consider the second term in the L.H.S. of (B.48).

$$\frac{\partial}{\partial S}\,\mathrm{tr}[D(S)P_{\theta\theta}\Lambda^T] = \frac{\partial}{\partial S}\,\mathrm{tr}[-GS(T_1+T_2+T_5+T_6)-G_\alpha S(T_3+T_4)],$$

$$= -[G^T(T_1+T_2+T_5+T_6)^T+G_\alpha^T(T_3+T_4)^T], \qquad (B.50)$$

where $T_1 = P^T_{\xi\hat{\xi}}\Lambda_{\xi\xi} + P_{\hat{\xi}\hat{\xi}}\Lambda^T_{\xi\hat{\xi}} + P_{\hat{\xi}x}\Lambda^T_{\xi x} + P_{\hat{\xi}\hat{x}}\Lambda^T_{\xi\hat{x}}$, (B.51)

$$T_2 = P^T_{\xi\hat{\xi}}\Lambda_{\xi\hat{\xi}} + P_{\hat{\xi}\hat{\xi}}\Lambda_{\hat{\xi}\hat{\xi}} + P_{\hat{\xi}x}\Lambda^T_{\hat{\xi}x} + P_{\hat{\xi}\hat{x}}\Lambda^T_{\hat{\xi}\hat{x}} \ , \qquad (B.52)$$

$$T_3 = P^T_{\xi\hat{x}}\Lambda_{\xi\xi} + P^T_{\hat{\xi}\hat{x}}\Lambda^T_{\xi\hat{\xi}} + P^T_{x\hat{x}}\Lambda^T_{\xi x} + P_{\hat{x}\hat{x}}\Lambda^T_{\xi\hat{x}} \ , \qquad (B.53)$$

$$T_4 = P^T_{\xi\hat{x}}\Lambda_{\xi\hat{\xi}} + P^T_{\hat{\xi}\hat{x}}\Lambda_{\hat{\xi}\hat{\xi}} + P^T_{x\hat{x}}\Lambda^T_{\hat{\xi}x} + P_{\hat{x}\hat{x}}\Lambda^T_{\hat{\xi}\hat{x}} \ , \qquad (B.54)$$

$$T_5 = P^T_{\xi\hat{x}}\Lambda_{\xi x} + P^T_{\hat{\xi}\hat{x}}\Lambda_{\hat{\xi}x} + P^T_{x\hat{x}}\Lambda_{xx} + P_{\hat{x}\hat{x}}\Lambda^T_{x\hat{x}} \ , \qquad (B.55)$$

$$T_6 = P^T_{\xi\hat{x}}\Lambda_{\xi\hat{x}} + P^T_{\hat{\xi}\hat{x}}\Lambda_{\hat{\xi}\hat{x}} + P^T_{x\hat{x}}\Lambda_{x\hat{x}} + P_{\hat{x}\hat{x}}\Lambda_{\hat{x}\hat{x}} \ . \qquad (B.56)$$

Next, consider the third term in the L.H.S. of (B.48).

$$\frac{\partial}{\partial S}\,\mathrm{tr}[P_{\theta\theta}D^T(S)\Lambda^T] = -\frac{\partial}{\partial S}\,\mathrm{tr}[\Lambda_{\xi\xi}GSP^T_{\xi\hat{\xi}}+\Lambda_{\xi\hat{\xi}}+GSP^T_{\xi\hat{\xi}}+\Lambda_{\xi x}GSP^T_{\xi\hat{x}} + \Lambda_{\xi\hat{x}}GSP^T_{\xi\hat{x}}$$

$$+ \Lambda^T_{\xi\hat{\xi}}GSP_{\hat{\xi}\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}}GSP_{\hat{\xi}\hat{\xi}} + \Lambda_{\hat{\xi}x}GSP^T_{\hat{\xi}\hat{x}} + \Lambda_{\hat{\xi}\hat{x}}GSP^T_{\hat{\xi}\hat{x}}$$

$$+ \Lambda^T_{\xi x}GSP_{\hat{\xi}x} + \Lambda^T_{\hat{\xi}x}GSP_{\hat{\xi}x} + \Lambda_{xx}GSP^T_{x\hat{x}} + \Lambda_{x\hat{x}}GSP^T_{x\hat{x}}$$

$$+ \Lambda^T_{\xi\hat{x}}GSP_{\hat{\xi}\hat{x}} + \Lambda^T_{\hat{\xi}\hat{x}}GSP_{\hat{\xi}\hat{x}} + \Lambda^T_{x\hat{x}}GSP_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}}GSP_{\hat{x}\hat{x}}]$$

$$- \frac{\partial}{\partial S}\mathrm{tr}[\Lambda_{\xi\xi}G_\alpha SP^T_{\xi\hat{x}} + \Lambda_{\xi\hat{\xi}}G_\alpha SP^T_{\xi\hat{x}} + \Lambda^T_{\xi\hat{\xi}}G_\alpha SP^T_{\hat{\xi}\hat{x}} + \Lambda_{\hat{\xi}\hat{\xi}}G_\alpha SP^T_{\hat{\xi}\hat{x}}$$

$$+ \Lambda^T_{\xi x}G_\alpha SP^T_{x\hat{x}} + \Lambda^T_{\hat{\xi}x}GSP^T_{x\hat{x}} + \Lambda^T_{\xi\hat{x}}G_\alpha SP_{\hat{x}\hat{x}} + \Lambda^T_{\hat{\xi}\hat{x}}G_\alpha SP_{\hat{x}\hat{x}}] \ ,$$

$$= -G^T(T_1 + T_2 + T_5 + T_6)^T - G_\alpha^T(T_3 + T_4)^T \qquad (B.57)$$

where $T_1, T_2, \ldots, T_6$ have been previously defined in (B.51) - (B.56).

Next, consider the fourth term in the L.H.S. of (B.48). Using (B.31), (B.38), Properties 3.1 and 3.2, one has

$$\frac{\partial}{\partial S} \operatorname{tr}[\Gamma(S)\Lambda^T] = 2\frac{\partial}{\partial S} \operatorname{tr}[M_2\{\underline{\mu}_\xi^T(\Lambda_{\xi\xi} + \Lambda_{\xi\hat{\xi}}^T + \Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})$$
$$+ \underline{x}_d^{*T}(\Lambda_{\xi x} + \Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}x} + \Lambda_{\hat{\xi}\hat{x}})^T\} + M_1\{\underline{\mu}_\xi^T(\Lambda_{\xi x} + \Lambda_{\hat{\xi}x} + \Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}}^T)$$
$$+ \underline{x}_d^{*T}(\Lambda_{xx} + \Lambda_{x\hat{x}}^T + \Lambda_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}})\}] . \qquad (B.58)$$

Using (B.20) and (B.21) in (B.58), one has

$$\frac{\partial}{\partial S} \operatorname{tr}[\Gamma(S)\Lambda^T] = 2G^T[(\Lambda_{\xi x} + \Lambda_{\hat{\xi}x} + \Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})^T\underline{\mu}_\xi + (\Lambda_{xx} + \Lambda_{x\hat{x}}$$
$$+ \Lambda_{x\hat{x}}^T + \Lambda_{\hat{x}\hat{x}})\underline{x}_d^*]\underline{x}_d^{*T} + 2G_\alpha^T[(\Lambda_{\xi\xi} + \Lambda_{\xi\hat{\xi}} + \Lambda_{\xi\hat{\xi}}^T + \Lambda_{\hat{\xi}\hat{\xi}})\underline{\mu}_\xi$$
$$+ (\Lambda_{\xi x} + \Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}x} + \Lambda_{\hat{\xi}\hat{x}})\underline{x}_d^*]\underline{x}_d^{*T} . \qquad (B.59)$$

Finally, it can be shown that the last two terms in the L.H.S. of (B.48) is given by

$$\frac{\partial}{\partial S} \operatorname{tr}[\underline{\lambda}\,\underline{\mu}^T D^T + \underline{\lambda}\,\underline{\bar{\beta}}^T] = -G^T[\underline{\mu}_\xi(\underline{\lambda}_\xi^T + \lambda_{\hat{\xi}}^T) + \underline{x}_d^*(\underline{\lambda}_x^T + \underline{\lambda}_{\hat{x}}^T) - (\underline{\lambda}_x + \underline{\lambda}_{\hat{x}})\underline{x}_d^{*T}]$$
$$-G_\alpha^T[\underline{x}_d^*(\underline{\lambda}_\xi^T + \underline{\lambda}_{\hat{\xi}}^T) - (\underline{\lambda}_\xi + \underline{\lambda}_{\hat{\xi}})\underline{x}_d^{*T}]. \qquad (B.60)$$

Hence, using (B.49), (B.50), and (B.57) - (B.59) in (B.48), one has the necessary condition,

$$2Q_2SP_{\xi\xi} - 2G^T[(\Lambda_{\xi\xi} + \Lambda^T_{\xi\hat{\xi}})P_{\xi\hat{\xi}} + (\Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})P_{\hat{\xi}\hat{\xi}} + (\Lambda_{\xi x} + \Lambda_{\hat{\xi}x})P^T_{\hat{\xi}x}$$

$$+ (\Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})P^T_{\hat{\xi}\hat{x}} + (\Lambda^T_{\xi x} + \Lambda^T_{\xi\hat{x}})(P_{\xi\hat{x}} - \underline{\mu}_\xi\underline{x}_d^{\star T})$$

$$+ (\Lambda^T_{\hat{\xi}x} + \Lambda^T_{\hat{\xi}\hat{x}})(P_{\hat{\xi}\hat{x}} - \underline{\mu}_\xi\underline{x}_d^{\star T}) + (\Lambda_{xx} + \Lambda^T_{x\hat{x}})(P_{x\hat{x}} - \underline{x}_d^\star\underline{x}_d^{\star T})$$

$$+ (\Lambda_{x\hat{x}} + \Lambda_{\hat{x}\hat{x}})(P_{\hat{x}\hat{x}} - \underline{x}_d^\star\underline{x}_d^{\star T})] - 2G_\alpha^T[(\Lambda_{\xi\xi} + \Lambda^T_{\xi\hat{\xi}})(P_{\xi\hat{x}} - \underline{\mu}_\xi\underline{x}_d^{\star T})$$

$$+ (\Lambda_{\xi\hat{\xi}} + \Lambda_{\hat{\xi}\hat{\xi}})(P_{\hat{\xi}\hat{x}} - \underline{\mu}_\xi\underline{x}_d^{\star T}) + (\Lambda_{\xi x} + \Lambda_{\hat{\xi}x})(P_{x\hat{x}} - \underline{x}_d^\star\underline{x}_d^{\star T})$$

$$+ (\Lambda_{\xi\hat{x}} + \Lambda_{\hat{\xi}\hat{x}})(P_{xx} - \underline{x}_d^\star\underline{x}_d^{\star T})] - G^T[\underline{\mu}_\xi(\underline{\lambda}_\xi^T + \underline{\lambda}_{\hat{\xi}}^T) + \underline{x}_d^\star(\underline{\lambda}_x^T + \underline{\lambda}_{\hat{x}}^T)$$

$$- (\underline{\lambda}_x + \underline{\lambda}_{\hat{x}})\underline{x}_d^{\star T}] - G_\alpha^T[\underline{x}_d^\star(\underline{\lambda}_\xi^T + \underline{\lambda}_{\hat{\xi}}^T) - (\underline{\lambda}_\xi + \underline{\lambda}_{\hat{\xi}})\underline{x}_d^{\star T}] = 0. \quad \text{(B.61)}$$

In summary, the auxiliary optimization problem defined by (B.1) - (B.6) is converted to its equivalent deterministic problem in the augmented space, defined by (B.12) and (B.14) for which the necessary conditions are obtained via the Matrix Minimum Principle. These necessary conditions are given by (B.42) - (B.47) and (B.60). The results of Properties 2.1 and 3.1 - 3.3 have been incorporated into these necessary conditions; this helps to simplify the results somewhat and it is only necessary to generate just $\underline{\mu}_\xi$ instead of all four of $\underline{\mu}_\xi$, $\underline{\mu}_x$, $\underline{\mu}_{\hat{\xi}}$, $\underline{\mu}_{\hat{x}}$.

APPENDIX C

DETAILS OF EXISTING COMPENSATION METHODS

A great deal of work has been done in the area of trajectory sensitivity compensation for LDR's as noted in Chapter I, but relatively little has been done for LSR's. This Appendix outlines the existing methods that compensate for the trajectory sensitivity of LSR's. These methods are neither based on trial and error nor the conventional methods using parameter tracking. They are based on the sensitivity concept which is very attractive in that the knowledge of the actual parameter values is not required. As far as the author is aware, only two of such methods have appeared in the literature: one by Higginbotham [35], and another by Stavroulakis and Sarachik [50].

Higginbotham's Compensation Method

Higginbotham [35] considers the LSR problem of finding a control $\underline{u}(t)$ such that the performance index

$$J = \frac{1}{2} E\{ ||\underline{x}(t_f)||^2_{S_f} + \int_{t_0}^{t_f} [||\underline{x}(t)||^2_{A(t)} + ||\underline{u}(t)||^2_{B(t)}]dt\} \quad (C.1)$$

minimized subject to

$$\dot{x} = F(\underline{\alpha},t)\underline{x} + G(\underline{\alpha},t)\underline{u} + \underline{w}(t); E\{\underline{x}(t_0)\} = \underline{x}_0, Var\{\underline{x}(t_0)\} = P_0 \ , \quad (C.2)$$

$$z = H(t)\underline{x}(t) + \underline{v}(t) \ , \quad (C.3)$$

where $\underline{x}(t)$ is a state n-vector; $\underline{u}$ is a control r-vector; $A(t)$ and S_f are symmetric positive semi-definite matrices; $B(t)$ is a symmetric positive definite matrix; $\underline{\alpha} = [\alpha_1\alpha_2\alpha_3....\alpha_m]^T$ is an actual parameter m-vector; t_o and t_f are the initial and final times which are fixed and specified, respectively; $\underline{w}(t)$ and $\underline{v}(t)$ are the process noise n-vector and the measurement noise ℓ-vector, respectively, with the following properties:

$$E\{\underline{w}(t)\underline{w}^T(\tau)\} = Q_w(t)\delta(t-\tau), \tag{C.4}$$

$$E\{\underline{v}(t)\underline{v}^T(\tau)\} = Q_v(t)\delta(t-\tau), \tag{C.5}$$

$$E\{\underline{w}(t)\} = E\{\underline{v}(t)\} = \underline{0}\ , \tag{C.6}$$

$$E\{\underline{w}(t)\underline{v}^T(\tau)\} = \underline{0}\ . \tag{C.7}$$

Higginbotham defines the trajectory sensitivity as the partial derivative of $\underline{x}(t)$ with respect to the actual parameter, $\underline{\alpha}$, i.e.,

$$\underline{\xi}_i \triangleq \frac{\partial \underline{x}}{\partial \alpha_i} \tag{C.8}$$

where ξ_i is the trajectory sensitivity corresponding to the actual parameter component, α_i. The differential equation for $\underline{\xi}_i$ is obtained by partially differentiating (C.2) with respect to α_i.

$$\dot{\underline{\xi}}_i = F(\underline{\alpha},t)\underline{\xi}_i + F_{\alpha_i}(\underline{\alpha},t)\underline{x}(t) + G_{\alpha_i}(\underline{\alpha},t)\underline{u}(t) + G(\underline{\alpha},t)\frac{\partial \underline{u}}{\partial \alpha_i},$$

$$\underline{\xi}_i(t_o) = \underline{0}, \; i = 1,2,\ldots,m, \tag{C.9}$$

where $\underline{x}(t)$ is the actual state, and the subscript α_i denotes a partial derivative with respect to α_i.

Then, in order to minimize simultaneously the state and the trajectory sensitivity, he looks for a control $\underline{u}(t)$, a function of both the state and the trajectory sensitivity, that minimizes the performance index

$$J_M = \frac{1}{2} E \{ ||\underline{x}(t_f)||^2_{A_1} + ||\underline{\xi}_a(t_f)||^2_{A_2} + \int^{t_f} [||\underline{x}(t)||^2_{A_3(t)} + \\ + ||\underline{\xi}_a||^2_{A_4(t)} + ||\underline{u}(t)||^2_{A_5(t)}] \, dt \}, \tag{C.10}$$

where $A_1, A_2, A_3(t)$ and $A_4(t)$ are symmetric positive semi-definite matrices; $A_5(t)$ is a symmetric positive definite matrix; and $\underline{\xi}_a(t)$ is an augmented trajectory sensitivity vector defined by

$$\underline{\xi}_a(t) \triangleq [\underline{\xi}_1^T(t) \; \underline{\xi}_2^T(t) \; \ldots . \; \underline{\xi}_m^T(t)]^T, \tag{C.11}$$

subject to (C.2) and (C.9). Now, since $\underline{u}(t)$ is a function of $\underline{x}(t)$ and $\underline{\xi}_a(t)$, one has

$$\frac{\partial \underline{u}(t)}{\partial \alpha_i} = \frac{\partial \underline{u}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \alpha_i} + \frac{\partial \underline{u}}{\partial \underline{\xi}_a} \cdot \frac{\partial \underline{\xi}_a}{\partial \alpha_i} ,$$

$$= \frac{\partial \underline{u}}{\partial \underline{x}} \cdot \underline{\xi}_i + \frac{\partial \underline{u}}{\partial \underline{\xi}_a} \cdot \frac{\partial \underline{\xi}_a}{\partial \alpha_i} . \tag{C.12}$$

It should be noted that $\frac{\partial \underline{\xi}_a}{\partial \alpha_i}$ is the second-order trajectory sensitivity. Substituting $\frac{\partial \underline{u}(t)}{\partial \alpha_i}$ from (C.12) in (C.9), one has the following:

$$\dot{\underline{\xi}}_i = [F(\underline{\alpha},t) + G(\underline{\alpha},t)\frac{\partial \underline{u}}{\partial \underline{x}}]\underline{\xi}_i + F_{\alpha_i}(\underline{\alpha},t)\underline{x}(t) + G_{\alpha_i}(\underline{\alpha},t)\underline{u}(t)$$

$$+ G(\underline{\alpha},t)\frac{\partial \underline{u}}{\partial \underline{\xi}_a}\frac{\partial \underline{\xi}_a}{\partial \alpha_i}. \qquad (C.13)$$

Since the actual parameter values, $\underline{\alpha}$, are unknown, it is not possible to evaluate $F(\underline{\alpha},t)$, $G(\underline{\alpha},t)$, $F_{\alpha_i}(\underline{\alpha},t)$, $G_{\alpha_i}(\underline{\alpha},t)$. Higginbotham assumes that these coefficient matrices are to be evaluated at the nominal parameter values, $\underline{\alpha}_n$. In addition, *instead of neglecting the second-order sensitivity term*, $\frac{\partial \underline{\xi}_a}{\partial \alpha_i}$, he replaces $\frac{\partial \underline{u}}{\partial \underline{\xi}_a} \cdot \frac{\partial \underline{\xi}_a}{\partial \alpha_i}$ by an r-vector $\underline{m}_i(t)$ which he refers to as the sensitivity control law. This results in a pseudo trajectory sensitivity, $\underline{\sigma}_i(t)$, which is given by

$$\dot{\underline{\sigma}}_i = [F(t) + G(t)\frac{\partial \underline{u}}{\partial \underline{x}}]\underline{\sigma}_i + F_{\alpha_i}(t)\underline{x}(t) + G_{\alpha_i}(t)\underline{u}(t) + G(t)\underline{m}_i(t)$$

$$i = 1,2,\ldots,m. \qquad (C.14)$$

The vector $\underline{\sigma}_i(t)$ is not the actual trajectory sensitivity of the system, whereas $\underline{\xi}_i(t)$ is. Higginbotham claims that $\underline{\sigma}_i(t)$ can be used as a measure of trajectory sensitivity in the sense that a reduction in $\underline{\sigma}_i(t)$ would result in corresponding reduction in the actual trajectory sensitivity, $\underline{\xi}_i(t)$. He, then, proceeds with the following formulation in an attempt to reduce simultaneously the state and the actual trajectory sensitivity.

$$\min_{\underline{u}_a(t)} J'_M = \min_{\underline{u}_a(t)} \frac{1}{2} E \{ ||\underline{z}_a(t_f)||^2_{A_6} + \int_{t_o}^{t_f} [||\underline{z}_a(t)||^2_{A_7(t)} + ||\underline{u}_a(t)||^2_{A_8(t)}] dt \}, \quad (C.15)$$

subject to

$$\dot{\underline{z}}_a = F_1(t)\underline{z}_a + G_1(t)\underline{u}_a + \underline{w}_1(t), \quad (C.16)$$

where

$$\underline{z}_a \triangleq \begin{bmatrix} \underline{x} \\ \\ \underline{\sigma}_a \end{bmatrix}, \qquad \underline{\sigma}_a = \begin{bmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_m \end{bmatrix};$$

$$\underline{u}_a \triangleq \begin{bmatrix} \underline{u} \\ \\ \underline{m}_a \end{bmatrix}, \qquad \underline{m}_a = \begin{bmatrix} \underline{m}_1 \\ \vdots \\ \underline{m}_m \end{bmatrix};$$

$$\underline{w}_1(t) \triangleq \begin{bmatrix} \underline{w} \\ \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \end{bmatrix};$$

$$F_1(t) \triangleq \begin{bmatrix} F & 0 & \cdots & 0 \\ F_{\alpha_1} & F_2 & \ddots & \vdots \\ F_{\alpha_3} & 0 & F_2 \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ F_{\alpha_m} & 0 & \cdots\ 0 & F_2 \end{bmatrix}, \quad F_2 \triangleq F(t) + G(t)\,\frac{\partial \underline{u}}{\partial \underline{x}}\,;$$

$$G_1(t) \triangleq \begin{bmatrix} G & 0 & \cdots & 0 \\ G_{\alpha_1} & G & \ddots & \vdots \\ G_{\alpha_2} & 0 & G \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ G_{\alpha_m} & 0 & \cdots\ 0 & G \end{bmatrix};$$

A_6 and $A_7(t)$ are symmetric positive semi-definite $(n + nm) \times (n + nm)$ matrices; and $A_8(t)$ is a symmetric positive definite $(r + mr) \times (r + mr)$ matrix; $\underline{z}_a$ is an augmented vector consisting of the state $\underline{x}(t)$ and the pseudo trajectory sensitivity vectors, $\underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_m$; $\underline{u}_a$ is an augmented control vector consisting of the state control law, $\underline{u}(t)$, and the sensitivity control laws: $\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_m$. Higginbotham, then, treats the minimization problem in the augmented space, defined by (C.15) and (C.16), as another LSR in which the Separation Theorem is applicable, i.e., the deterministic controller can be designed independently of the Kalman filter.

Let $C_a(t)$ denote the gain of the deterministic controller of the augmented system. It can be shown that $C_a(t)$ can be obtained from (C.17) and (C.18) below.

$$C_a(t) = A_8^{-1}(t)G_1^T(t)S_a(t) \ , \tag{C.17}$$

$$\dot{S}_a(t) = -S_a(t)F_1(t) - F_1^T(t)S_a(t) + C_a^T(t)A_8(t) - A_7(t). \tag{C.18}$$

And the optimal augmented feedback control, $\underline{u}_a(t)$, is given by

$$\underline{u}_a(t) = -C_a(t)\underline{z}_a(t) \ . \tag{C.19}$$

By reason that $\underline{x}(t)$ is not available because of being corrupted with noise, Higginbotham replaces $\underline{x}(t)$ in (C.19) by its estimate, $\hat{\underline{x}}(t)$; and hence

$$\underline{u}_a(t) = -C_a(t)\hat{\underline{z}}_a(t) \ , \tag{C.20}$$

where
$$\hat{\underline{z}}_a(t) \triangleq [\hat{\underline{x}}^T(t) \ \underline{\sigma}_1^T \ \underline{\sigma}_2^T \ldots \underline{\sigma}_m^T]. \tag{C.21}$$

Higginbotham does not at all account for the modelling errors in the Kalman filter; he uses the standard equations exactly the same as (2.27) - (2.29) to generate the estimate, $\hat{\underline{x}}(t)$, for use in (C.20). Equation (C.20) can be expanded in terms of sub-vectors and sub-matrices as

$$\underline{u}(t) = -C_{a11}(t)\hat{\underline{x}}(t) - C_{a12}\underline{\sigma}_a, \tag{C.22}$$

$$\underline{m}_a(t) = -C_{a22}(t)\underline{\sigma}_a(t) - C_{a21}\hat{\underline{x}}, \tag{C.23}$$

$$C_a(t) \triangleq \begin{bmatrix} C_{a11} & C_{a12} \\ C_{a21} & C_{a22} \end{bmatrix}, \tag{C.24}$$

where C_{a11}, C_{a12}, C_{a21} and C_{a12} are $r \times n$, $r \times nm$, $mr \times n$ and $mr \times mn$ sub-matrices, respectively. Higginbotham refers to $\underline{u}(t)$ as the state control law, and $\underline{m}_a(t)$ as the sensitivity control law. The state control law, $\underline{u}(t)$, is generated by using the estimate, $\hat{\underline{x}}(t)$, from the Kalman filter, and the sensitivity control law, $\underline{m}_a(t)$, by using (C.14) with $x(t)$ replaced by $\underline{x}(t)$ and $\frac{\partial \underline{u}}{\partial \underline{x}} = -C_{a11}$ to generate $\underline{\sigma}_i(t)$, $i=1,2,...,m$, on-line.

In summary, in his compensation method, Higginbotham considers and compensates for only the deterministic part, i.e., the deterministic controller, of the LSR, while leaving the Kalman filter untouched. The number of equations involved in this method is

(a) $\frac{(n+mn)}{2}(1+n+nm)$ non-linear Riccati type of equations for an off-line solution of $S_a(t)$ from (C.18);

(b) $\frac{n}{2}(n+1)$ non-linear Riccati type of equations for the an off-line solution of the error covariance matrix $P(t)$, see (2.29); and

(c) $(n+mn)$ equations for on-line solutions of $\hat{\underline{x}}(t)$ and $\underline{\sigma}_i(t)$, $i = 1,2,3,...,m$. Hence, the total number of off-line equations is $\frac{1}{2}[(2+2m+m^2)n^2 + (2+m)n]$ the total number of on-line equations is $(n+mn)$ equations, and the net total is $\frac{1}{2}[(2+2m+m^2)n^2 + (4+3m)n]$ equations.

The underlying assumptions used either explicitly or implicitly by Higginbotham are as follows.

(h_1) The parameter errors, $\delta\underline{\alpha} \overset{\Delta}{=} \underline{\alpha} - \underline{\alpha}_n$, are small so that the use of first order sensitivity terms is sufficiently accurate;

(h_2) The controller gain of the augmented system, C_a, can be designed independently of the Kalman filter; and

(h_3) The pseudo trajectory sensitivity, $\underline{\sigma}_i(t)$, $i = 1,2,...,m$, can be used as a measure of the actual trajectory sensitivity, $\underline{\xi}_i$, $i = 1,2,...,m$.

Stavroulakis and Sarachik's Compensation Method

Stavroulakis and Sarachik consider an LSR problem similar to that defined by (C.1) - (C.7) with a scalar actual parameter α. In order to minimize simultaneously the state and trajectory sensitivity, they look for an output feedback control

$$\underline{u}(t) = D_2(t)\hat{\underline{x}}(t) \quad , \tag{C.25}$$

such that the cost

$$J_k = \frac{1}{2} E\{\int_{t_o}^{t_f} [||\underline{x}||^2_{B_1(t)} + ||\underline{u}||^2_{B_2(t)} + ||\underline{\xi}||^2_{B_3(t)}]dt\}, \tag{C.26}$$

is minimized, where $B_1(t)$ and $B_3(t)$ are symmetric positive semi-definite $n \times n$ matrices, $B_2(t)$ is a symmetric positive definite $r \times r$ matrix, and the trajectory sensitivity $\underline{\xi}(t)$ is defined by

$$\underline{\xi}(t) = \frac{\partial \underline{x}(t)}{\partial \alpha}\bigg|_{\alpha=\alpha_n} , \tag{C.27}$$

in which α_n is the scalar nominal parameter. $\hat{\underline{x}}(t)$, an estimate of the state $\underline{x}(t)$, is the output of the Kalman filter defined by (2.27) - (2.29). The state $\underline{x}(t)$ and its estimate $\hat{\underline{x}}(t)$ are related by

$$\underline{x}(t) = \hat{\underline{x}}(t) + \underline{e}(t) , \tag{C.28}$$

where $\underline{e}(t)$ is the estimation error. Substituting $\underline{x}(t)$ from (C.28) in (C.26) results in

$$J_k = \frac{1}{2} \{\int_{t_o}^{t_f} [\, ||\hat{\underline{x}}||^2_{B_1(t)} + ||\underline{u}(t)||^2_{B_2(t)} + ||\underline{\xi}||^2_{B_3(t)}]dt\} + \frac{1}{2} E\{\int_{t_o}^{t_f} ||\underline{e}||^2_Q \, dt\} , \tag{C.29}$$

due to the orthogonality of $\hat{\underline{x}}(t)$ and $\underline{e}(t)$. Also, the differential equation for $\underline{e}(t)$ (see [43] for details) is found to be independent of the input $\underline{u}(t)$ so that minimizing (C.29) is equivalent to minimizing

$$J_k' = \frac{1}{2} E\{ \int_{t_o}^{t_f} [\, ||\hat{\underline{x}}(t)||^2_{B_1(t)} + ||\underline{u}(t)||^2_{B_2(t)} + ||\underline{\xi}(t)||^2_{B_3(t)}] \, dt\} . \tag{C.30}$$

Next, without mathematical justifications, the authors replace $\underline{\xi}(t)$ in the last term of the integrand of (C.30) by the state-estimate sensitivity, $\hat{\underline{\xi}}(t)$, which is defined as

$$\underline{\hat{\xi}}(t) = \left.\frac{\partial \underline{\hat{x}}(\alpha,t)}{\partial \alpha}\right|_{\alpha=\alpha_n} . \tag{C.31}$$

Hence, equation (C.30) now becomes

$$\hat{J}_k = \frac{1}{2} E \{\int_{t_o}^{t_f} [\,||\hat{x}||^2_{B_1(t)} + ||\underline{u}(t)||^2_{B_2(t)} + ||\underline{\hat{\xi}}(t)||_{B_3(t)}]dt\} \tag{C.32}$$

Next, they derive the differential equation for $\underline{\hat{\xi}}(t)$ by partially differentiating (2.21) with respect to α. Here, they treat the term,

$$\underline{\eta}(t) \overset{\Delta}{=} \underline{z}(t) - H(t)\underline{\hat{x}}(\alpha,t), \tag{C.33}$$

which is known as the residual or the innovation process, as being independent of the parameter α, and mistakenly arrive at the result

$$\left.\frac{\partial \underline{\eta}(t)}{\partial \alpha}\right|_{\alpha=\alpha_n} = \frac{\partial}{\partial \alpha}[\underline{z}(t) - H(t)\underline{\hat{x}}(\alpha,t)]\,|_{\alpha=\alpha_n}, \tag{C.34}$$

$$= 0 .$$

Property C.1 at the end of this appendix will show that in general $\frac{\partial \underline{\eta}(t)}{\partial \alpha} \neq 0$. The correct differential equation for $\underline{\hat{\xi}}(t)$ is as follows.

$$\dot{\underline{\hat{\xi}}} = [F(t)+G(t)D_2(t)-K(t)H(t)]\underline{\hat{\xi}} + K(t)H(t)\underline{\xi} + [F_\alpha(t)+G_\alpha(t)D_2(t)$$

$$-K_\alpha(t)H(t)]\underline{\hat{x}} + K_\alpha(t)H(t)\underline{x} + K_\alpha(t)\underline{v}(t), \quad \underline{\hat{\xi}}(t_o) = \underline{0} \tag{C.35}$$

Also, the differential equation for $\dot{\underline{\xi}}(t)$ is obtained by partially differentiating (C.2) with respect to α.

$$\dot{\underline{\xi}} = G(t)D_2(t)\underline{\xi} + F(t)\underline{\xi} + F_\alpha(t)\underline{x} + G_\alpha(t)D_2(t)\hat{\underline{x}}, \quad \underline{\xi}(t_0) = \underline{0}. \tag{C.36}$$

Therefore, the correct final form of the compensation method attempted by Stavroulakis and Sarachik is given in the following.

$$\min_{D_2(t)} \hat{J}_k = \min_{D_2(t)} \frac{1}{2} E\{\int_{t_0}^{t_f} [\|\hat{\underline{x}}(t)\|^2_{B_1(t)} + \|\underline{u}(t)\|^2_{B_2(t)} + \|\hat{\underline{\xi}}\|^2_{B_3(t)}]dt\}, \tag{C.37}$$

subject to (C.35) and (C.36). The authors then proceed in a fashion similar to the compensation method of this dissertation to convert (C.37) into its deterministic equivalent,

$$\min_{D_2(t)} \hat{J}_k = \min_{D_2(t)} \frac{1}{2} \mathrm{tr}\{\int_{t_0}^{t_f} [B_1(t)P_{\hat{x}\hat{x}}(t) + D_2^T(t)B_2(t)D_2(t)P_{\hat{x}\hat{x}}(t) + B_3(t)P_{\hat{\xi}\hat{\xi}}(t)]dt\}, \tag{C.38}$$

in which $\underline{u}(t)$ from (C.25) has been substituted in, and the definition similar to (B.22) has also been used. The optimization problem above can be expressed in a compact form in terms of the augmented state $\underline{\theta}(t)$ which is defined according to (B.11).

It can similarly be shown that in the augmented space the optimization problem becomes

$$\min_{D_2(t)} \hat{J}_k = \min_{D_2(t)} \frac{1}{2}\,\mathrm{tr}\,\{\int_{t_o}^{t_f} [T(t)+W(t)]P_{\theta\theta}(t)dt\},\quad (C.39)$$

subject to

$$\dot{P}_{\theta\theta} = D_3(t)P_{\theta\theta} + P_{\theta\theta}D^T(t) + E_3(t) \qquad (C.40)$$

where

$$D_3(t) \triangleq \begin{bmatrix} F & GD_2 & F_\alpha & G_\alpha D_2 \\ KH & F+GD_2-KH & K_\alpha H & F_\alpha+G_\alpha D_2-K_\alpha H \\ 0 & 0 & F & GD_2 \\ 0 & 0 & KH & F+GD_2-KH \end{bmatrix},$$

$$E_3(t) \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_\alpha Q_v K_\alpha^T & 0 & K_\alpha Q_v K^T \\ 0 & 0 & Q_w & 0 \\ 0 & KQ_v K_\alpha^T & 0 & KQ_v K^T \end{bmatrix},$$

$$T \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (B_1+D_2^T B_2 D_2) \end{bmatrix},$$

$$
W \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .
$$

This is of the same form as the previous optimization problem defined by (B.35), (B.32) and (B.24). When the Matrix Minimum Principle is applied, it results in the necessary conditions similar to those of the previous optimization problem. It would involve $2n(4n+1)$ equations for the state $P_{\theta\theta}$, and $2n(4n+1)$ equations for the costate Λ for a scalar parameter. Stavroulakis and Sarachik did not consider a multi-parameter case in [50]. However, the extension to the case where $\underline{\alpha} = [\alpha_1 \alpha_2 \ldots \alpha_m]^T$ is obvious and it is given as follows. Equations (C.39) and (C.40) become

$$
\min_{D_2(t)} J_k = \min_{D_2(t)} \frac{1}{2} \operatorname{tr} \{ \int_{t_o}^{t_f} [B_1(t) + D_2^T(t) B_2(t) D_2(t)] P_{\hat{x}\hat{x}}(t) dt + \int_{t_o}^{t_f} \sum_{i=1}^{m} W^i(t) P_{\theta\theta}^i(t) dt \}, \tag{C.41}
$$

subject to

$$
\dot{P}_{\theta\theta}^i = D_3^i(t) P_{\theta\theta}^i(t) + P_{\theta\theta}^i(t) D^{i^T}(t) + E^i(t), \quad i = 1,2,\ldots,m, \tag{C.42}
$$

where

$$
\underline{\xi}_i(t) \triangleq \left. \frac{\partial \underline{x}}{\partial \alpha_i} \right|_{\underline{\alpha} = \underline{\alpha}_n}, \quad \hat{\underline{\xi}}_i(t) \triangleq \left. \frac{\partial \hat{\underline{x}}}{\partial \alpha_i} \right|_{\underline{\alpha} = \underline{\alpha}_n}, \tag{C.43}
$$

$$\underline{\theta}_i(t) \triangleq [\underline{\xi}_i^T \quad \hat{\underline{\xi}}_i^T \quad \underline{x}^T \quad \hat{\underline{x}}^T]^T , \tag{C.44}$$

$$P_{\theta\theta}^i(t) \triangleq E\{\underline{\theta}_i(t)\underline{\theta}_i^T(t)\} , \tag{C.45}$$

$$W^i(t) \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_3^i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C.46}$$

$$D_3^i(t) \triangleq \begin{bmatrix} F & GD_2 & F_{\alpha_i} & G_{\alpha_i}D_2 \\ KH & F+GD_2-KH & K_{\alpha_i}H & F_{\alpha_i}+G_{\alpha_i}D_2-K_{\alpha_i}H \\ 0 & 0 & F & GD_2 \\ 0 & 0 & KH & F+GD_2-KH \end{bmatrix}, \tag{C.47}$$

$$E_3^i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_{\alpha_i}Q_vK_{\alpha_i}^T & 0 & K_{\alpha_i}Q_vK^T \\ 0 & 0 & Q_w & 0 \\ 0 & KQ_vK_{\alpha_i}^T & 0 & KQ_vK^T \end{bmatrix}. \tag{C.48}$$

It can be shown that when the Matrix Minimum Principle is applied to the above multi-parameter, multi-variable optimization problem it will result in a TPBVP involving in $(6m+2)n^2 + (m+1)n$ state equations, $2nm(4n+1)$ costate equations. It also requires another $\frac{mn}{2}(n+1)$ equations for $K_{\alpha_i}(t)$, $i=1,2,...,m$; and hence the net total is

$\frac{1}{2}[(29m+4)n^2 + (7m+2)n]$ equations.

In summary, the correct version of the compensation method by Stavroulakis and Sarachik is a refinement over Higginbotham's compensation method in that they consider the modelling errors in both the Kalman filter and, in effect, the deterministic controller.

The essence of the method is to find an output feedback control such that the cost $\hat{J}_k$, which provides for a trade-off between the state estimate, the state-estimate sensitivity and the control effort, is minimized. The assumptions employed either explicitly or implicitly by them are as follows:

(s_1) The parameter variations, $\delta\underline{\alpha} \triangleq \underline{\alpha} - \underline{\alpha}_n$, are assumed to be small;

(s_2) Replacement of $\underline{\xi}(t)$ in the integrand of J_k' in (C.30) by $\hat{\underline{\xi}}(t)$ would still result in a reduction in the trajectory sensitivity, $\underline{\xi}(t)$.

<u>Property C.1</u>: It is not true in general that $\left.\frac{\partial\underline{\eta}(t)}{\partial\alpha}\right|_{\alpha=\alpha_n} = 0$ for all t, where $\underline{\eta}(t)$, known as the innovation process or the residual, is defined by

$$\underline{\eta}(t) \triangleq \underline{z}(t) - H(t)\hat{\underline{x}}(\alpha,t) \ . \qquad \text{(C.49)}$$

<u>Proof</u>:

If it was true that $\left.\frac{\partial\underline{\eta}(t)}{\partial\alpha}\right|_{\alpha=\alpha_n} = \underline{0}$ for all t, then one would have from (C.49),

$$\frac{\partial}{\partial\alpha}[\underline{z}(t)-H(t)\hat{\underline{x}}(\alpha,t)]_{\alpha=\alpha_n} = \underline{0}, \text{ for all } t \qquad (C.50)$$

Substituting $\underline{z}(t)$ from (C.3) in (C.50), one would have

$$\frac{\partial}{\partial\alpha}[H(t)\underline{x}(\alpha,t) + \underline{v}(t) - H(t)\hat{\underline{x}}(\alpha,t)]_{\alpha=\alpha_n} = \underline{0}, \text{ for all } t. \qquad (C.51)$$

$$H(t)\left.\frac{\partial\underline{x}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n} - H(t)\left.\frac{\partial\hat{\underline{x}}(\alpha,t)}{\partial\alpha}\right|_{\alpha=\alpha_n} = \underline{0}, \text{ for all } t. \qquad (C.52)$$

Using (C.27) and (C.31),

$$H(t)[\underline{\xi}(t) - \hat{\underline{\xi}}(t)] = \underline{0}, \quad \text{for all } t. \qquad (C.53)$$

Since in general H(t) is not a zero matrix, one would have from (C.53)

$$\underline{\xi}(t) = \hat{\xi}(t), \text{ for all } t. \qquad (C.54)$$

In the following, it will be proved that the relationship in (C.54) is not true in general; and hence, in general, $\left.\frac{\partial\underline{n}(t)}{\partial\alpha}\right|_{\alpha=\alpha_n} \neq 0$, for all t. The plant under consideration is given by (C.2) and the state estimate is given by (2.21). Subtracting (C.2) from (2.21), with $\alpha=\alpha_n$ in both equations,

$$\dot{\underline{\ell}}_x = (F-KH)\underline{\ell}_x + K\underline{v} - \underline{w}, \quad \underline{\ell}_x(t_0) = \underline{0}, \qquad (C.55)$$

where
$$\underline{\ell}_x \overset{\Delta}{=} \hat{\underline{x}} - \underline{x} \ . \qquad (C.56)$$

Similarly, subtracting (C.36) from (C.35), one has

$$\dot{\underline{\ell}}_\xi = (F-KH)\underline{\ell}_\xi + (F_\alpha - K_\alpha H)\underline{\ell}_x + K_\alpha \underline{v}, \quad \underline{\ell}_\xi(t_o) = \underline{0}, \tag{C.57}$$

$$\underline{\ell}_\xi \triangleq \hat{\underline{\xi}} - \underline{\xi} \; . \tag{C.58}$$

Equations (C.55) and (C.57) can be expressed as

$$\dot{\underline{\ell}}_z = D_4(t)\underline{\ell}_z + \underline{\beta}_4(t), \quad \underline{\ell}_z(t_o) = [\underline{0}^T \; \underline{0}^T \; \underline{0}^T \; \underline{0}^T]^T, \tag{C.59}$$

where
$$\underline{\ell}_z \triangleq [\underline{\ell}_\xi^T \quad \underline{\ell}_x^T]^T \; , \tag{C.60}$$

$$D_4(t) \triangleq \begin{bmatrix} F-KH & F_\alpha - K_\alpha H \\ 0 & F-KH \end{bmatrix}, \tag{C.61}$$

$$\beta_4(t) \triangleq [(K_\alpha V)^T \; (K\underline{v}-\underline{w})^T]^T \, . \tag{C.62}$$

The solution of (C.59) is given by

$$\underline{\ell}_z(t) = \int_{t_o}^{t} \Phi_\ell(t,\alpha)\underline{\beta}_4(t)d\alpha \; ,$$

where $\Phi_\ell(t,\alpha)$ is the transition matrix of (C.59).

$$\underline{\ell}_z = \int_{t_o}^{t} \Phi_\ell(t,\alpha) \begin{bmatrix} K_\alpha(\alpha)\underline{v}(\alpha) \\ K(\alpha)\underline{v}(\alpha) - \underline{w}(\alpha) \end{bmatrix} d\alpha \tag{C.63}$$

Since the integrand of (C.63) is not equal to zero in general,

$$\underline{\ell}_z(t) \neq \underline{0}, \quad \text{for all } t.$$

Hence,

$$\underline{\ell}_\xi(t) \neq 0, \text{ for all } t.$$

That is, $$\underline{\xi}(t) \neq \hat{\underline{\xi}}(t), \text{ for all } t. \qquad (C.64)$$

Therefore, it can be concluded that

$$\left.\frac{\partial \underline{\eta}(t)}{\partial \alpha}\right|_{\alpha=\alpha_n} \neq 0, \text{ for all } t.$$

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